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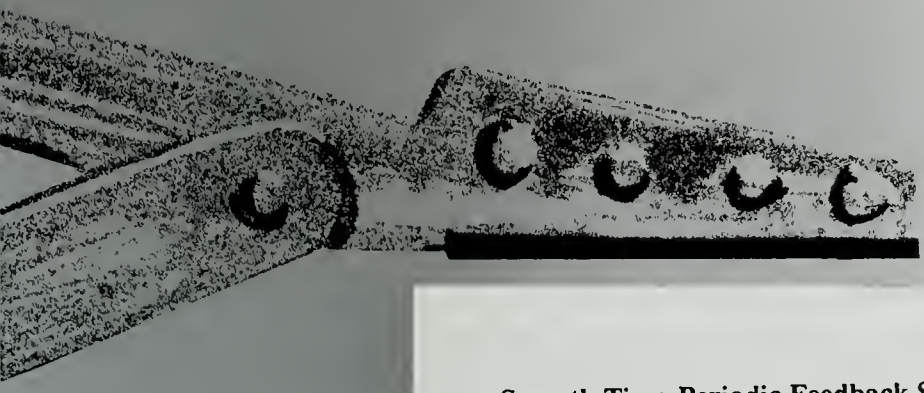

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Technical Report No. 598
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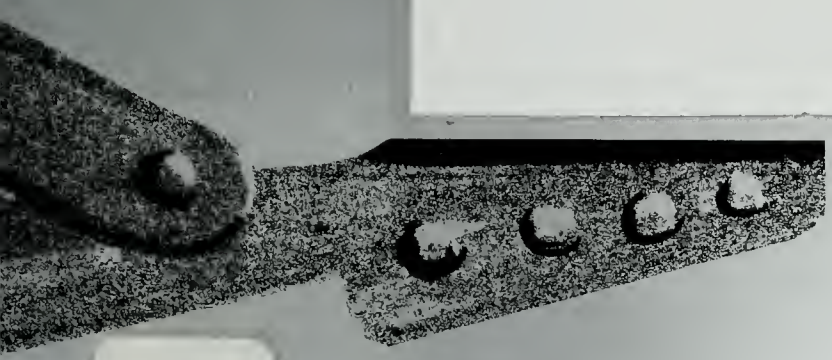

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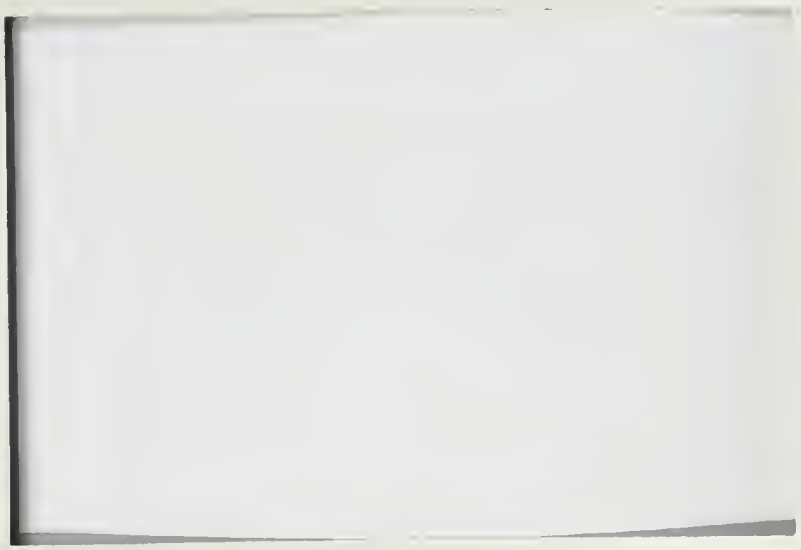

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Smooth Time-Periodic Feedback Solutions for Nonholonomic Motion Planning

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Abstract

In this paper, we present an algorithm for computing time-periodic feedback solutions for nonholonomic motion planning with collision-avoidance. For a first-order Lie bracket system, we begin by computing a holonomic collision-free path using the potential field method. Then, we compute a nonholonomic path approximating the collision-free path within a predetermined bound. For this we first solve for extended inputs of an extended system using Lie bracket completion vectors. We then use averaging techniques to calculate the asymptotic trajectory of the nonholonomic system under application of a family of highly-oscillatory inputs. Comparing the limiting trajectories with the extended system we obtain a system of nonlinear equations from which the desired admissible control inputs can be solved. For higher-order Lie bracket systems we use multi-scale averaging and apply recursively the algorithm for first-order Lie bracket systems. Based on averaging techniques we also provide error bounds between a nonholonomic system and its averaged system.

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1 Introduction

Nonholonomic Motion Planning (NMP) is the study of path planning for a robotic system subject to either nonholonomic constraints, e.g., rolling constraint, or nonintegrable conservation laws, e.g., conservation of angular momentum. NMP arises in dextrous manipulation with a multifingered robotic hand ([FGL91b], [Li89] and [MS90]), path planning for mobile robots([Lau92] and [JC89]), control of nonholonomic actuators ([Bro88]), and reconfiguration of a space structure using internal motion ([FGL91b], [Kri90], [NM90] and [PD90]).

The problem of nonholonomic motion planning can be stated as follows:

Problem 1.1 *(***Nonholonomic Motion Planning Problem:***) Let Q be a n-dimensional configuration space of a robotic system, x_0 and x_f two given initial and final configurations, $B(x) \in \Re^{n\times m}$ a m-dimensional completely nonholonomic distribution*[1] *(or constraints) which translates into a system of differential equations of the form*

$$\dot{x} = b_1(x)u_1 + \cdots + b_m(x)u_m \;=\; B(x)u \tag{1}$$

and

$$C_i(x) \leq 0, i = 1, \cdots l \tag{2}$$

a set of position constraints for collision avoidance. Compute a set of input $u(t) \in \Re^m$, $t \in [0,T]$, perhaps of optimal cost, such that the resulting trajectory $x(t) \in Q, t \in [0,T]$, is collision-free and links x_f to x_0.

Research in nonholonomic motion planning has expanded recently. For example, Brockett ([Bro81] and [BD91]) and Murray and Sastry ([MS90]

[1] This is equivalent to the nonlinear control system (1) being controllable.

and [MS92]) studied analytic solutions of a family of canonical or chained systems using Fourier analysis and optimal control. They applied feedback transformation and linearization techniques to reduce a general system to the canonical or chained form. Lafferriere and Sussmann ([LS90]) studied analytic solutions of a class of nilpotent or nilpotenizable systems. Their iterative algorithm can also produce approximate solutions for non-nilpotent or non-nilpotentizable systems. In ([Mon89], [Mon92], [BM90] and [RMB92]), optimal solutions and feedback control for coupled rigid-body systems or nonholonomic systems with drift were studied. Based on Ritz approximation theory, Fernandes, Gurvits and Li ([FGL91b] and [FGL91a]) gave a numerical algorithm for NMP along with some interesting simulation results. In [SL91], Sussmann and Liu presented a new approach to nonholonomic motion planning with collision-avoidance.

It is clear that the approaches taken by ([Bro81], [BD91], [MS90], [MS92], [LS90] and [FGL91b]) can be classified as open-loop. While there are certain advantages associated with each of these open-loop approaches, an open-loop approach is less likely to produce solutions which are robust with respect to modeling uncertainties and senoring errors, as compared with a feedback approach. However, as Brockett pointed out ([Bro83]) that smooth, time-independent feedbacks do not exist in general for a nonholonomic motion planning system.

In view of Brockett's result and the fact that a typical robotic task demands real-time generation of nonholonomic paths, we will present in this paper a new algorithm for computing collision-free, smooth time-periodic feedback solutions for NMP systems. The algorithm is efficient and requires only ordinary computational tools for its implementation. In addition, through the use of averaging techniques we will offer a geometrical interpretation to the approach used in ([SL91]). Here, the different time-scales of averaging arise because motion in the directions of the control vector fields is much faster than that in the directions of the Lie-bracket vector fields.

Furthermore, if a system under consideration is of higher Lie-bracket order, then multi-scale averaging results and the feedback algorithm becomes recursive.

The paper is organized as follows: In Section 2.1, we present the algorithm for a first-order Lie bracket system along with a simple example. In Section 2.2, we generalize the algorithm to higher-order Lie bracket systems. In Section 2.3, we apply averaging techniques to prove two important theorems used in the feedback algorithm and give estimates for error bounds. In Section 3, we provide motivations for the control inputs used in the feedback algorithm by studying the control of a class of nonholonomic systems. In Section 3.2, we present some controllability results of a linearized nonholonomic system around a nominal trajectory. In Appendix A, we compare some of our results with that of Coron's and establish a negative result on the existence of smooth time-periodic feedbacks for exponential stability. In Appendix B, we study analytic solutions for generalized Brockett system.

2 Averaging and the Feedback Algorithm

In this section, we describe an algorithm that computes time-periodic feedback solutions for NMP. In contrast to the Basis Algorithm ([FGL91b]), this algorithm will give solutions which are

- collision-free and
- robust with respect to modeling uncertainties and sensoring errors.

Furthermore, only ordinary computational tools will be needed for its implementation.

We will first consider the case of first-order Lie bracket systems and then generalize the results to higher-order Lie bracket systems.

2.1 The First-Order Lie Bracket Case

Consider the following controllable nonholonomic system

$$\dot{x} = b_1(x)u_1 + b_2(x)u_2 + \cdots + b_m(x)u_m = B(x)u, \quad x \in \Re^n \tag{3}$$

with initial and final configurations $x_0, x_f \in \Re^n, (n > m)$. The system is said to be a first-order Lie bracket system if only first-order Lie bracket operations are necessary to generate the controllability Lie algebra. For example, let $l \triangleq m(m-1)/2$ then the rank of the following Lie algebra completion matrix

$$\{b_1(x), b_2(x), \cdots b_m(x), c_1(x), \cdots c_l(x)\} \triangleq C(x)$$

is n, $\forall x \in \Re^n$, where

$$c_i(x) = [b_j(x), b_k(x)] = \frac{\partial b_k(x)}{\partial x} b_j(x) - \frac{\partial b_j(x)}{\partial x} b_k(x)$$

denotes the Lie bracket vector field of b_j and b_k.

The algorithm that computes time-periodic feedback solutions for system (3) has four steps and is described below. We leave the proof of correctness to Section 2.3.

Algorithm 2.1 (Feedback Algorithm for 1st-Order Systems)

Step 1 *Computing Collision-free Holonomic Path:* Use holonomic motion planning techniques (see, e.g., [SS86], [Can88], [Yap87], [Kod87] and [Rim90]) to construct a **collision-free** (holonomic) dynamical system

$$\dot{y} = g(y), \; y(0) = x_0 \in \Re^n \tag{4}$$

such that the goal configuration, x_f, subject to some topological considerations, is essentially the global attractor. For example, if we follow the potential field method as in ([Kod87] and [Rim90]) then $g(y)$ is the gradient vector field of some navigation function.

Other than practical considerations dynamical system (4) can be chosen arbitrary. Our objective is to find for system (3) time-periodic feedbacks of the form

$$u = u(x, t, \epsilon)$$

where ϵ is a small parameter, such that the C-norm of the trajectory error between the holonomic system (4) and the nonholonomic system (3) goes to zero as $\epsilon \to 0$,

$$\lim_{\epsilon \to 0} \|x - y\|_{C[0,T]} \to 0$$

Here, T is fixed and the $C-$norm of a continuous function $y(t) \in \Re^n, t \in [0, T]$, is defined by

$$\|y\|_{C[0,T]} = \max_{t \in [0,T]} |y(t)|$$

Step 2 Compute extended inputs $v = (v_1, v_2, \cdots v_{m+l})^T$ from the following system of linear equations

$$(b_1, \cdots b_m, c_1, \cdots c_l)\, v = C(y)v = g(y)$$

Since the Lie bracket completion matrix $C(y)$ has rank n, the solution is given by

$$v = C^T(CC^T)^{-1}g(y) \triangleq C^\sharp g(y)$$

Thus, the collision-free holonomic path satisfies the following system of differential equations

$$\dot{y} = b_1 v_1 + b_2 v_2 + \cdots + b_m v_m + c_1 v_{m+1} + \cdots + c_l v_{m+l} \qquad (5)$$

with initial condition $y(0) = x_0$.

Step 3 Let the control input of the nonholonomic system (3) be of the form

$$u_i^\epsilon = \alpha_i(x) + \sqrt{\frac{2}{\epsilon}} \sum_{j=1}^{l} \beta_i^j(x) \sin(jt/\epsilon) + \gamma_i^j(x) \cos(jt/\epsilon) \qquad (6)$$

where ϵ is a small parameter.

Clearly, u_i^ϵ is time-periodic feedback. It becomes highly oscillatory as $\epsilon \to 0$. The asymptotic behavior of system (3) under application of control input (6) can be computed using averaging techniques ([Arn78]). Following the results of Section 2.3 (see also [Gur91]) the asymptotic trajectory satisfies a differential equation of the form

$$\dot{y} = \sum_{i=1}^{m} \delta_i(y) b_i + \sum_{i<j} \eta_{ij} [b_i, b_j] \tag{7}$$

where

$$\delta_i = \alpha_i + \sum_{j=1}^{m} \sum_{k=1}^{l} \left(\langle \nabla \beta_i^k,\, b_j \rangle \gamma_j^k - \langle \nabla \gamma_i^k,\, b_j \rangle \beta_j^k \right) \operatorname{sgn}(j-i) \frac{1}{k}$$

$$\operatorname{Skew}(\eta_{ij}) \triangleq \begin{pmatrix} 0 & \eta_{12} & \eta_{13} & \cdot & \cdot & \cdot \\ -\eta_{12} & 0 & \eta_{23} & \cdot & \cdot & \cdot \\ -\eta_{13} & -\eta_{23} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \sum_{k=1}^{l} \left(\beta^k \wedge \gamma^k \right) \frac{1}{k}$$

$$\beta^k = (\beta_1^k, \cdots, \beta_m^k)^T, \;\; \gamma^k = (\gamma_1^k, \cdots, \gamma_m^k)^T$$

and the $\wedge$ (wedge) product of two vectors $x, y \in \Re^n$ gives a $n \times n$ skew-symmetric matrix

$$x \wedge y = xy^T - yx^T.$$

∇f denotes the gradient vector field of the function f.

Step 4 *Solving for the Desirable Inputs:* Equating the coefficients of system (5) with that of (7) yield a system of nonlinear equations

$$\sum_{k=1}^{l} \left(\beta^k \wedge \gamma^k \right) \frac{1}{k} = \operatorname{Skew}(\eta_{ij}) \tag{8}$$

$$\alpha_i + \sum_{j=1}^{m} \sum_{k=1}^{l} \left(\langle \nabla \beta_i^k,\, b_j \rangle \gamma_j^k - \langle \nabla \gamma_i^k,\, b_j \rangle \beta_j^k \right) \operatorname{sgn}(j-i) \frac{1}{k} = v_i, \tag{9}$$

where $i = 1, \cdots m$. Solving for (α, β, γ) from Eqs. (8) and (9) gives the desirable inputs to the nonholonomic system.

Remark 2.1 (1) Estimates of the trajectory errors between the holonomic system and the nonholonomic system can be found in Section 2.3. (2) The algorithm also applies when both systems are non-autonomous, i.e.,

$$\begin{aligned} \dot{x} &= B(x,t)u, \\ \dot{y} &= g(y,t) \end{aligned}$$

In this case, the Lie bracket and gradient operations are performed for fixed time t, and the control input has the form

$$u_i = \alpha_i(x,t) + \sqrt{\frac{2}{\epsilon}} \sum_{j=1}^{l} \beta_i^j(x,t) \sin(\frac{jt}{\epsilon}) + \gamma_i^j(x,t) \cos(\frac{jt}{\epsilon}).$$

□

Example 2.1 We use the *rolling disk* example to illustrate the *feedback algorithm*. Let (x, y) be the contact location, ϕ the angle and (u_1, u_2) the steering and rolling velocity, respectively, of a rolling disk. Then, the equations describing the constraint are given by (see [FGL91b])

$$\begin{bmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \cos\phi \\ \sin\phi \end{bmatrix} u_2 \triangleq b_1 u_1 + b_2 u_2 \tag{10}$$

It is clear that this is a first-order Lie bracket system as the completion matrix

$$C = (b_1, b_2, [b_1, b_2]) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

is nonsingular over the configuration space.

Assuming that the holonomic system is given by the following exponentially stable system

$$\dot{z} = -z \tag{11}$$

We want to construct inputs for system (10) whose asymptotic behavior as $\epsilon \to 0$ is given by that of (11).

From Step 2, the extended inputs are found to be

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = C^T(CC^T)^{-1} \begin{bmatrix} -\phi \\ -x \\ -y \end{bmatrix} = \begin{bmatrix} -\phi \\ -(x\cos\phi + y\sin\phi) \\ x\sin\phi - y\cos\phi \end{bmatrix}$$

Choose the following forms of the control inputs

$$\begin{aligned} u_1^\epsilon &= \alpha_1 + \sqrt{\frac{2}{\epsilon}}\beta_1 \cos\frac{t}{\epsilon} \\ u_2^\epsilon &= \alpha_2 + \sqrt{\frac{2}{\epsilon}}\beta_2 \sin\frac{t}{\epsilon} \end{aligned}$$

where we have set γ_i's to zero for simplicity.

Thus, the system of nonlinear equations from Steps 3 and 4 have the form

$$\begin{aligned} \alpha_1 - \beta_2\langle\nabla\beta_1, b_2\rangle &= -\phi \\ \alpha_2 + \beta_1\langle\nabla\beta_2, b_1\rangle &= -(x\cos\phi + y\sin\phi) \\ \beta_1\beta_2 &= x\sin\phi - y\cos\phi \end{aligned} \tag{12}$$

It is not difficult to check that

$$\begin{aligned} \beta_1 &= 1, \alpha_1 = -\phi \\ \beta_2 &= x\sin\phi - y\cos\phi \\ \alpha_2 &= -2(x\cos\phi + y\sin\phi) \end{aligned}$$

constitute a set of solution to Eq. (12). Thus, the time-periodic feedback solutions are

$$\begin{aligned} u_1 &= -\phi + \sqrt{\frac{2}{\epsilon}}\cos(\frac{t}{\epsilon}) \triangleq u_1(\phi, t, \epsilon) \\ u_2 &= -2(x\cos\phi + y\sin\phi) + \sqrt{\frac{2}{\epsilon}}(x\sin\phi - y\cos\phi)\sin(\frac{t}{\epsilon}) \\ &\triangleq u_2(\phi, x, y, t, \epsilon) \end{aligned}$$

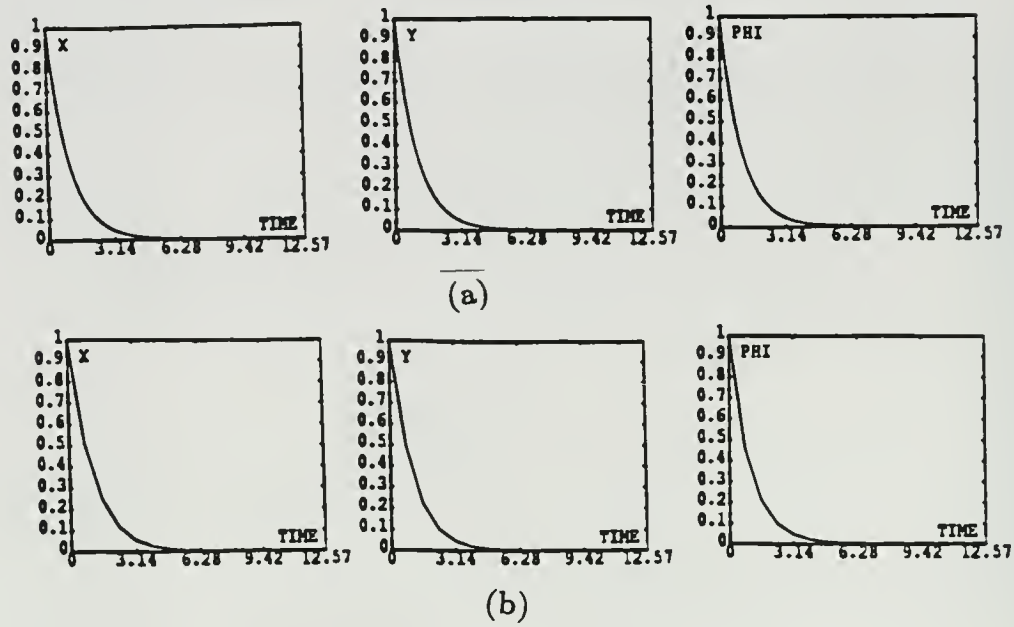

Figure 1: Simulated trajectories for (a) $\epsilon = 0.0001$ and (b) $\epsilon = 0.015625$.

The solution grows out of bound as $\epsilon \to 0$. To avoid unbounded control, we can use symmetry of a driftless system to rescale the control input

$$\begin{aligned} \tilde{u}_1(\phi, t, \epsilon) &= \sqrt{\epsilon} u_1(\phi, t\sqrt{\epsilon}, \epsilon) \\ \tilde{u}_2 &= \sqrt{\epsilon} u_2(\phi, x, y, t\sqrt{\epsilon}, \epsilon) \end{aligned}$$

Let $\tilde{x}(t) \in \Re^3$ be the corresponding nonholonomic trajectory, we have that

$$x(t\sqrt{\epsilon}) = \tilde{x}(t)$$

Figure 1 show simulation results for two different values of ϵ and scaled control inputs $\tilde{u}_i$. □

Remark 2.2 (1) The feedback input constructed above does not vanish at the origin, as $u_1(0, t) \neq 0$, while the control input in [Cor91] is identically zero at the origin. (2) It is impossible to find (β_1, β_2) which are differentiable at the origin and satisfy $\beta_1(0) = \beta_2(0) = 0$. If so, from the equation

$$\beta_1 \beta_2 = x \sin\phi - y \cos\phi$$

we have that

$$\frac{\partial}{\partial y}(\beta_1\beta_2)|_0 = \beta_2\frac{\partial\beta_1}{\partial y} + \beta_1\frac{\partial\beta_2}{\partial y} = -\cos 0 = -1$$

resulting in a contradiction. □

It is interesting to note that solutions of the nonholonomic system converge to a small attractor containing the origin. Denote by B a ball of the origin. Then, solutions of the nonholonomic system have the form

$$x(t) = e^{-t}x(0) + \sqrt{\epsilon}G(x,t), \quad t \in [0, 2\pi]$$

for all $x_0 \in B$, where $G(x,t)$ is bounded. For sufficiently small ϵ, $x(2\pi)$ is also in B. We can use $x(2\pi)$ as the new initial condition and apply the same set of control inputs to the system. Consequently, trajectories of the nonholnomic system converge to a subset of a ball centered at the origin with radius

$$\sqrt{\epsilon}\max_{x\in B, t\in[0,2\pi]} G(x,t).$$

2.2 The Higher-Order Lie Bracket Case

To compute time-periodic feedback solutions for higher-order Lie bracket systems we introduce multiple small parameters and use a recursive algorithm.

Consider a r^{th}-order Lie bracket system,

$$\dot{x} = b_1u_1 + \cdots + b_mu_m, \quad x \in \Re^n \tag{13}$$

together with the following holonomic system

$$\dot{y} = g(y), \quad y(0) = x(0) \in \Re^n \tag{14}$$

and time interval $[0, 2\pi]$.

Algorithm 2.2 (Recursive Algorithm for Higher-Order Systems)

Step 1 Compute a Lie bracket completion matrix for the system

$$C(x) = \{b_1, \cdots b_m, [b_i, b_j], \cdots \underbrace{[b_{i_1}, [b_{i_2}, [\cdots, b_{i_r}] \cdots]}_{r}\} \tag{15}$$

Step 2 Compute the extended inputs by the formula

$$(v_1, \cdots v_m, v_{m+1}, \cdots v_{m+\eta})^T = C^\sharp \cdot g(x).$$

where $C^\sharp = C^T(CC^T)^{-1}$ is the generalized inverse of C and η is the number of entries of $C(x)$ less m.

Step 3 Let $r > 1$ be the order of $C(x)$, we form another matrix $C_l(x)$ of order no more than $(r-1)$ as follows. First, rearrange the entries of $C(x)$ so that

$$C(x) = \{\bar{C}_l(x), C_h(x)\} \tag{16}$$

where $\bar{C}_l(x)$ consists of vectors of order strictly less than r and $C_h(x)$ consists of the highest order vectors. Second, let $\tilde{C}_l(x)$ be a minimal set of vector fields such that its order is strictly less than r and any vector in $C_h(x)$ can be obtained by a single Lie bracket operation on the entries of $\bar{C}_l(x)$ and $\tilde{C}_l(x)$. Finally, set

$$C_l(x) = \{\bar{C}_l(x), \tilde{C}_l(x)\} \triangleq \{\underbrace{b_1, \cdots b_m, f_1, \cdots f_{s_1}}_{\bar{C}_l(x)}, \underbrace{g_1, \cdots g_{s_2}}_{\tilde{C}_l(x)}\}. \tag{17}$$

It is clear from the above construction that, using Algorithm 2.1, trajectories of the system defined by $C(x)$ can be approximated by that of the system defined by $C_l(x)$.

Step 4 Introduce a small parameter ϵ_1 and use Algorithm 2.1 to compute the coefficients of the following input functions

$$\begin{aligned} u_i^{\epsilon_1} &= \gamma_i(x,t) + \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{\rho} \alpha_{k,1}^i(x,t) \sin(\frac{kt}{\epsilon_1}) + \beta_{k,1}^i(x,t) \cos(\frac{kt}{\epsilon_1}), \\ v_i^{\epsilon_1} &= \mu_i(x,t) + \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{\rho} \alpha_{k,2}^i(x,t) \sin(\frac{kt}{\epsilon_1}) + \beta_{k,2}^i(x,t) \cos(\frac{kt}{\epsilon_1}), \end{aligned}$$

$$w_i^{\epsilon_1} = \frac{1}{\sqrt{\epsilon}} \sum_{k=1}^{\rho} \alpha_{k,3}^i(x,t) \sin(\frac{kt}{\epsilon_1}) + \beta_{k,3}^i(x,t) \cos(\frac{kt}{\epsilon_1})$$

where the index i runs from 1 to m for the first equation, 1 to s_1 for the second equation and 1 to s_2 for the last equation, and ρ is defined as follows: First, let X be the minimal set of vector fields such that any element of $C_h(x)$ is the Lie bracket vector field of two vectors in X. Then, denote by p the number of entries of X, and we have

$$\rho = p(p-1)/2.$$

Step 5 If the Lie bracket order of $C_l(x)$ is no less than 1, then replace $C(x)$ with $C_l(x)$ and repeat Step 3 and 4, using another small parameter and the extended inputs computed previously.

Note that the asymptotic expression for the error function in the k^{th}-step of the above recursive algorithm can be shown to be (see Section 2.3)

$$\begin{aligned} \Delta_0 &= 0, \\ \Delta_{s+1} &= \sqrt{\epsilon_{s+1}} \left(C_1 K_s + e^{2\pi C_3 K_s} C_2 K_s^2 \right) \qquad (18) \\ K_0 &= \max_{i,k,x} \{ \|f_i(x)\|, \|G_k(x)\| \}, \\ K_{s+1} &= \frac{1}{\sqrt{\epsilon_{s+1}}} C_4 K_s, \qquad (19) \end{aligned}$$

and the accumulated error is

$$\Delta = \|x - y\|_{C[0,2\pi]} = \Delta_1 + \cdots + \Delta_r.$$

In order for the nonholonomic trajectories to converge to that of the nominal system, the ϵ_s's have to be chosen so that Δ is within an acceptable bound.

Example 2.2 The constraint equations of a ball rolling on a plane have the form

$$\dot{x} = \begin{bmatrix} 0 \\ sec x_1 \\ -\sin x_5 \\ -\cos x_5 \\ -tan x_1 \end{bmatrix} u_1 + \begin{bmatrix} -1 \\ 0 \\ -\cos x_5 \\ \sin x_5 \\ 0 \end{bmatrix} u_2 \triangleq b_1(x)u_1 + b_2(x)u_2$$

Let $b_3 = [b_1, b_2]$, $b_4 = [b_1, [b_1, b_2]] = [b_1, b_3]$ and $b_5 = [b_2, b_3]$, it is not difficult to check that (see, e.g., [LC90])

$$C(x) = \{b_1, b_2, b_3, b_4, b_5\}$$

defines a Lie-bracket completion of the system and the Lie-bracket order is 2.

Initially, Step 3 of the algorithm gives

$$\bar{C}_l(x) = \{b_1, b_2, b_3\}$$

$$C_h(x) = \{b_4, b_5\},$$

$\tilde{C}_l(x)$ is empty and $\rho = 3$. □

Example 2.3 Consider another 2-inputs and 3-states system of the form

$$\dot{x} = b_1(x)u_1 + b_2(x)u_2 \tag{20}$$

Assume that the Lie-bracket completion matrix has the form

$$C(x) = \{b_1, b_2, [[b_1, b_2], [b_1, [b_1, b_2]]]\}$$

In other words, this is an order 3 system.

Initially, we have that

$$\bar{C}_l(x) = \{b_1, b_2\},\ C_h(x) = \{[[b_1, b_2], [b_1, [b_1, b_2]]]\}$$

$$\tilde{C}_l(x) = \{[b_1, b_2], [b_1, [b_1, b_2]]\}$$

and $\rho = 1$. Thus, the nominal system is given by

$$\dot{y} = b_1 v_1 + b_2 v_2 + [[b_1, b_2], [b_1, [b_1, b_2]]]v_3 \tag{21}$$

and the *nonholonomic system* by

$$\dot{x} = b_1 w_1 + b_2 w_2 + [b_1, b_2]w_3 + [b_1, [b_1, b_2]]w_4 \tag{22}$$

Applying Step 4 of the algorithm we can find the w_i's in terms of the v_i's.

$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_2 \\ w_3 &= \sqrt{\frac{2}{\epsilon_1}}\cos(\frac{t}{\epsilon_1})v_3 \\ w_4 &= \sqrt{\frac{2}{\epsilon_1}}\sin(\frac{t}{\epsilon_1}) \end{aligned}$$

Then, the new *nominal system* is given by (22) while the *nonholonomic system* by

$$\dot{x} = b_1 r_1 + b_2 r_2 + [b_1, b_2] r_3. \tag{23}$$

Introduce another small parameter ϵ_2 and solving for the $r_i's$ in terms of the $w_i's$ we get

$$\begin{aligned} r_1 &= w_1 + \sqrt{\frac{2}{\epsilon_2}}\cos(\frac{t}{\epsilon_2}) = v_1 + \sqrt{\frac{2}{\epsilon_2}}\cos(\frac{t}{\epsilon_2}) \\ r_2 &= w_2 = v_2 \\ r_3 &= w_3 + \sqrt{\frac{2}{\epsilon_2}}\cos(\frac{t}{\epsilon_2})w_4 \\ &= \sqrt{\frac{2}{\epsilon_1}}\cos(\frac{t}{\epsilon_1})v_3 + \sqrt{\frac{2}{\epsilon_2}}\cos(\frac{2}{\epsilon_2})\sqrt{\frac{2}{\epsilon_1}}\sin(\frac{t}{\epsilon_1}) \end{aligned}$$

Finally, with *nominal system* given by (23) and the nonholonomic system given by (20) the true control inputs are found to be

$$\begin{aligned} u_1 &= v_1 + \sqrt{\frac{2}{\epsilon_2}}\cos(\frac{t}{\epsilon_2}) + \sqrt{\frac{2}{\epsilon_3}}\cos(\frac{t}{\epsilon_3}) \\ u_2 &= v_2 + \sqrt{\frac{2}{\epsilon_3}}\sin(\frac{t}{\epsilon_3})\left(\sqrt{\frac{2}{\epsilon_1}}\cos(\frac{t}{\epsilon_1})v_3 + \sqrt{\frac{2}{\epsilon_2}}\cos(\frac{t}{\epsilon_2})\sqrt{\frac{2}{\epsilon_1}}\sin(\frac{t}{\epsilon_1})\right) \end{aligned}$$

□

2.3 Averaging and Asymptotic Behaviour

In this section, we use averaging techniques to compute asymptotic behaviour of a nonholonomic system under application of some highly oscillatory inputs.

Consider the following nonholonomic system

$$\dot{x} = \sum_{i=1}^{m} b_i(x) u_i^{\epsilon} \tag{24}$$

with control input of the form

$$u_i^{\epsilon} = \bar{u}_i(t) + \frac{1}{\sqrt{\epsilon}} \tilde{u}_i(t, \theta) \tag{25}$$

where $\theta = t/\epsilon$, and the function $\tilde{u}_i$ is periodic in θ with period 2π, and has zero average

$$\int_0^{2\pi} \tilde{u}_i(t, \theta) d\theta = 0. \tag{26}$$

We can identify θ as the *fast* variable, and (x, t) the *slow* variables. Using the Averaging Theorem ([Arn78], pp. 293-297) we can, for small enough ϵ, replace the above system with an "averaged" system for which the fast variable is eliminated. Moreover, the averaged system serves as a good approximation of system (24).

Theorem 2.1 *For sufficiently small ϵ, the trajectory of system (24) is bounded by solution of the following system*

$$\dot{z} = \sum_{i=1}^{m} b_i(z) \bar{u}_i + \frac{1}{2\pi} \sum_{i<j} [b_i, b_j] \nu_{i,j}, \quad z(0) = x(0) = x_0 \tag{27}$$

in the sense that

$$\|x - z\|_{C[0,2\pi]} \triangleq \Delta_\epsilon \leq \Delta_{1,\epsilon} + \Delta_{2,\epsilon}$$

where

$$\nu_{i,j} = \int_0^{2\pi} \int_0^{\theta} \tilde{u}_i(t, \tau) \tilde{u}_j(t, \theta) d\tau d\theta \tag{28}$$

and $(\Delta_{1,\epsilon}, \Delta_{2,\epsilon})$ are parameters which tend to zero as $\epsilon \to 0$.

In other words, system (27) defines the asymptotic behaviour of system (24) as $\epsilon \to 0$.

Proof. The derivatives of the slow variables with respect to the fast variable are seen to be

$$\begin{aligned} x' &\triangleq \frac{dx}{d\theta} = \epsilon \sum_{i=1}^{m} b_i(x)\bar{u}_i(t) + \sqrt{\epsilon}\sum_{i=1}^{m} b_i(x)\tilde{u}_i(t,\theta) \\ t' &= \epsilon \end{aligned} \tag{29}$$

Introduce new variable $y \in \Re^n$ by

$$x = y + \sqrt{\epsilon}\phi_1(y,t,\theta) + \epsilon\phi_2(y,t,\theta) + \cdots \tag{30}$$

where the function (ϕ_1, ϕ_2) are $2\pi-$periodic in θ and are chosen so that y satisfies a simpler differential equation

$$y' = \frac{dy}{d\theta} = \epsilon\,(f_1(y,t) + \sqrt{\epsilon}f_2(y,t) + \cdots) \tag{31}$$

We can find the functions $(\phi_1, \phi_2, f_1, f_2, \cdots)$ by first differentiating (30) with respect to θ,

$$\begin{aligned} x' &= y' + \sqrt{\epsilon}(\frac{\partial\phi_1}{\partial y}y' + \epsilon\frac{\partial\phi_1}{\partial t} + \frac{\partial\phi_1}{\partial\theta}) + \epsilon(\frac{\partial\phi_2}{\partial y}y' + \epsilon\frac{\partial\phi_2}{\partial t} + \frac{\partial\phi_2}{\partial\theta}) + \cdots \\ &= \sqrt{\epsilon}\frac{\partial\phi_1}{\partial\theta} + \epsilon\frac{\partial\phi_2}{\partial\theta} + \epsilon^{3/2}\frac{\partial\phi_1}{\partial t} + \epsilon^2\frac{\partial\phi_2}{\partial t} + (1 + \sqrt{\epsilon}\frac{\partial\phi_1}{\partial y} + \epsilon\frac{\partial\phi_2}{\partial y})y' \\ &= \sqrt{\epsilon}\frac{\partial\phi_1}{\partial\theta} + \epsilon\frac{\partial\phi_2}{\partial\theta} + \epsilon^{3/2}\frac{\partial\phi_1}{\partial t} + \epsilon^2\frac{\partial\phi_2}{\partial t} + \cdots \\ &+ (1 + \sqrt{\epsilon}\frac{\partial\phi_1}{\partial y} + \epsilon\frac{\partial\phi_2}{\partial y})(\epsilon f_1(y,t) + \epsilon^{3/2}f_2(y,t) + \cdots) + \cdots \end{aligned} \tag{32}$$

and then computing the Taylor series expansion of (29) using (30)

$$\begin{aligned} x' &= \epsilon\sum_{i=1}^{m} b_i(y)\bar{u}_i(t) + \sqrt{\epsilon}\sum_{i=1}^{m} b_i(y)\tilde{u}_i(t,\theta) \\ &+ \epsilon\sum_{i=1}^{m}\frac{\partial b_i}{\partial y}(\sqrt{\epsilon}\phi_1 + \epsilon\phi_2 + \cdots)\bar{u}_i + \sqrt{\epsilon}\sum_{i=1}^{m}\frac{\partial b_i}{\partial y}(\sqrt{\epsilon}\phi_1 + \epsilon\phi_2 + \cdots)\tilde{u}_i \\ &+ \epsilon^{3/2}\sum_{i=1}^{m}\sum_{k=1}^{n}\langle\frac{\partial^2 b_i^k}{\partial^2 y}\phi_1,\ \phi_1\rangle e_k\tilde{u}_i + O(\epsilon^2) \end{aligned} \tag{33}$$

where

$$b_i = \sum_{k=1}^{n} b_i^k e_k,$$

and $\{e_k\}_{k=1}^n$ is the canonical basis of $\Re^n$.

Finally, equating the coefficients of $\epsilon^{1/2}$ from Eqs. (32) and (33) we have that

$$\frac{\partial \phi_1}{\partial \theta} = \sum_{i=1}^{m} b_i(y)\tilde{u}_i(t,\theta) \tag{34}$$

The solution of Eq. (34) is given by

$$\phi_1(y,t,\theta) = \sum_{i=1}^{k} b_i(y) \int_0^\theta \tilde{u}_i(t,\tau)d\tau + \bar{\phi}_1(y,t).$$

Note that $\phi_1(y,t,\theta)$ is 2π-periodic in θ because of (26).

Equating the coefficients of ϵ we have that

$$\sum_{i=1}^{m} b_i(y)\bar{u}_i(t) + \sum_{i=1}^{m} \frac{\partial b_i}{\partial y}\phi_1\tilde{u}_i - \frac{\partial \phi_2}{\partial \theta} = f_1(y,t) \tag{35}$$

Integrating both sides of (35) over 2π and taking into account that $f_1(y,t)$ is independent of θ yield

$$\begin{aligned}
f_1(y,t) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=1}^{m} b_i(y)\bar{u}_i(t) + \sum_{i=1}^{m}\frac{\partial b_i}{\partial y}\tilde{u}_i(t,\theta)\phi_1(y,t,\theta) - \frac{\partial \phi_2}{\partial \theta}\right)d\theta \\
&= \sum_{i=1}^{m} b_i(y)\bar{u}_i(t) \\
&+ \frac{1}{2\pi}\sum_{i=1}^{m}\int_0^{2\pi}\frac{\partial b_i}{\partial y}\tilde{u}_i(t,\theta)\left(\sum_{j}^{m} b_j\int_0^\theta \tilde{u}_j(t,\tau)d\tau + \bar{\phi}_1(y,t)\right)d\theta \\
&= \sum_{i=1}^{m} b_i(y)\bar{u}_i(t) + \frac{1}{2\pi}\sum_i\sum_j \frac{\partial b_i}{\partial y}b_j\int_0^{2\pi}\int_0^\theta \tilde{u}_j(t,\tau)d\tau\tilde{u}_i(t,\theta)d\theta \\
&= \sum_{i=1}^{m} b_i(y)\bar{u}_i(t) + \frac{1}{2\pi}\sum_{i<j}[b_i,b_j]\nu_{i,j}
\end{aligned}$$

where

$$\nu_{i,j} = \int_0^{2\pi}\int_0^\theta \tilde{u}_i(t,\tau)\tilde{u}_j(t,\theta)d\tau d\theta \tag{36}$$

In the above derivation, we have made use of the facts that ϕ_2 is 2π-periodic in θ, and $\tilde{u}_i(t,\theta)$ has zero average (26). The last equality follows from integration by parts.

Using the formula for $f_1(y,t)$ we have for ϕ_2 that

$$\begin{aligned}\frac{\partial \phi_2}{\partial \theta} &= \sum_i^m b_i(y)\bar{u}_i(t) + \sum_i^m \frac{\partial b_i}{\partial y}\tilde{u}_i(t,\theta)\phi_1(y,t,\theta) - f_1(y,t) \\ &= \sum_i^m \frac{\partial b_i}{\partial y}\tilde{u}_i(t,\theta)\phi_1(y,t,\theta) - \sum_{i<j}[b_i,b_j]\nu_{i,j}\end{aligned}$$

which gives

$$\phi_2 = \int_0^\theta \left(\sum_i^m \frac{\partial b_i}{\partial y}\tilde{u}_i(t,\tau)\phi_1(y,t,\tau) - \sum_{i<j}[b_i,b_j]\nu_{i,j} \right) d\tau + \bar{\phi}_2(y,t).$$

Equating the coefficients of $\epsilon^{3/2}$ we have that

$$\sum_i^m \frac{\partial b_i}{\partial y}\phi_1\bar{u}_i + \sum_i^m \frac{\partial b_i}{\partial y}\phi_2\tilde{u}_i(t,\theta) + \sum_i^m \sum_{k=1}^n \langle \frac{\partial^2 f_i^k}{\partial^2 y}\phi_1, \phi_1\rangle = f_2(y,t) + \frac{\partial \phi_1}{\partial t}$$

Integrating the above equation with respect to θ from 0 to 2π yields

$$\begin{aligned}f_2(y,t) &= \frac{1}{2\pi}\int_0^{2\pi} \sum_i^m \frac{\partial b_i}{\partial y}(\phi_1\bar{u}_i + \phi_2\tilde{u}_i)\, d\theta \\ &+ \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_i^m \sum_{k=1}^n \tilde{u}_i \langle \frac{\partial^2 b_i^k}{\partial^2 y}\phi_1, \phi_1\rangle e_k - \frac{\partial \phi_1}{\partial t} \right) d\theta \qquad (37)\end{aligned}$$

We choose $\bar{\phi}_1(y,t)$ to be

$$\bar{\phi}_1(y,t) = -\int_0^{2\pi} \sum_i^m b_i(y) \int_0^{2\pi} \tilde{u}_i(t,\tau) d\tau d\theta$$

so that $\phi_1(y,t,\theta)$ has zero average,

$$\int_0^{2\pi} \phi_1(y,t,\theta) d\theta = 0$$

Using the above property of ϕ_1 in Eq. (37) we have

$$f_2(y,t) = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_i^k \frac{\partial b_i}{\partial y}\phi_2\tilde{u}_i(t,\theta) + \sum_i^m \sum_{k=1}^n \tilde{u}_i \langle \frac{\partial^2 b_i^k}{\partial^2 y}\phi_1, \phi_1\rangle e_k \right) d\theta$$

Finally, we have the following form for the derivative of y with respect to the slow variable t

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{\epsilon}\frac{dy}{d\theta} = f_1(y,t) + \epsilon^{1/2} f_2(y,t) + \cdots \\ &= \sum_{i=1}^{m} b_i(y)\bar{u}_i + \frac{1}{2\pi}\sum_{i<j}[b_i,b_j]\nu_{i,j} + \epsilon^{1/2} f_2(y,t) + O(\epsilon) \qquad (38)\end{aligned}$$

and

$$x(t) = y + \sqrt{\epsilon}\phi_1 + \epsilon\phi_2 + \cdots$$

with $x(0) = y(0) = x_0$.

Define the averaged system by

$$\dot{z} = \sum_{i=1}^{m} b_i(z)\bar{u}_i(t) + \frac{1}{2\pi}\sum_{i<j}[b_i,b_j]\nu_{i,j}, \quad z(0) = y(0) = x(0). \qquad (39)$$

The error between the averaged system and the original system can be computed using techniques from ([Arn78]) and is given by

$$\Delta_\epsilon = \sup_{t\in[0,2\pi]} \|x(t) - z(t)\| \leq \Delta_{1,\epsilon} + \Delta_{2,\epsilon}$$

where

$$\Delta_{1,\epsilon} = \|x - y\|_{C[0,2\pi]}, \ \Delta_2 = \|z - y\|_{C[0,2\pi]}$$

For sufficiently small ϵ, we also have

$$\begin{aligned}\Delta_{1,\epsilon} &\leq \sqrt{\epsilon}\|\phi_1\| + O(\epsilon) \\ \Delta_{2,\epsilon} &\leq \sqrt{\epsilon}(e^{2\pi K}/K)\|f_2\|_{C[0,2\pi]}\end{aligned}$$

where

$$K = \sum_{i=1}^{m} \|\frac{\partial b_i}{\partial y}\| \cdot \|\bar{u}_i\| + \sum_{i<j} \|\frac{\partial [b_i,b_j]}{\partial y}\| \cdot \|\nu_{i,j}\|$$

and all norms are the maximum norm defined in some neighborhood of $z(t)$.
□

In view of Eq. (28) the problem is to find, for given $\nu_{i,j}, (i < j)$, 2π-periodic function $\tilde{u}_j(t,\theta)$ which has zero average and satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} \tilde{u}_j(t,\theta)\int_0^{\theta} \tilde{u}_i(t,\tau)d\tau d\theta = \frac{1}{2\pi}\nu_{i,j}(t) \tag{40}$$

The solution to the problem, using Fourier method of separation of variables, has the form

$$\tilde{u}_i(t,\theta) = \sum_{p=1}^{k} \alpha_p^i(t)\sin p\theta + \beta_p^i(t)\cos p\theta \tag{41}$$

Define

$$\alpha_p = (\alpha_p^1, \cdots \alpha_p^m)^T, \quad \beta_p = (\beta_p^1, \cdots \beta_p^m)^T$$

It is easy to check, through direct integration, that

$$\text{Skew}(\nu_{i,j}) = \pi \sum_{p=1}^{k} \frac{1}{p}\beta_p \wedge \alpha_p \tag{42}$$

where

$$\beta_p \wedge \alpha_p = \beta_p \alpha_p^T - \alpha_p \beta_p^T.$$

In other words, we have to solve the following *factorization* problem: For a given skew-symmetric matrix S, find β_p, α_p such that

$$S = \sum_{p=1}^{k} \beta_p \wedge \alpha_p \tag{43}$$

In the case $k = m(m-1)/2$, the solution is given by

$$\text{Skew}(v_{i,j}) = \sum_{1 \le i < j \le m} (\nu_{i,j} e_i \wedge e_j)$$

Remark 2.3 Here the elements of S, β_p and α_p belong to some commutative algebra X over $\Re$. In our context it is the functional algebra. In other words, we are solving nonholonomic motion planning problem for the following infinite dimensional system

$$\begin{aligned} \dot{x} &= u, \ x(0) = x(2\pi), \\ \dot{S} &= x \wedge u, \ S(2\pi) - S(0) = \text{Skew}(\nu_{i,j}) \end{aligned}$$

using finite Fourier representation for the control input

$$u = \sum_p \alpha_p \sin pt + \beta_p \cos pt$$

This motivates one to study the canonical systems introduced by R. Brockett ([Bro81]). In fact, the same ideas can be applied to the family of chained systems of higher Lie bracket orders, with states and control inputs belonging to an arbitrary commutative algebra over $\Re$. For example, the algorithm given in [MS90] works for any commutative algebra.

If the algebra is the reals or the complex numbers, we can easily find out the minimal k solving Eq. (43), namely

$$\min(k) = \text{rank}(S)/2$$

and the solution is given by the so-called symplectic decomposition of S. In the general case the problem is difficult to solve.

Proposition 2.1 *Assume the following input to system (29)*

$$u_i = \bar{u}_i(x) + \sum_{k=1}^{m(m-1)/2} \alpha_i^k(x) \sin(k\frac{t}{\epsilon}) + \beta_i^k(x) \cos(k\frac{t}{\epsilon}) \tag{44}$$

then the asymptotic trajectory is defined by solution of system (45)

$$\dot{z} = \sum_{i=1}^{m} \delta_i b_i(x) + \sum_{1 \le i < j \le m} [b_i, b_j]\nu_{i,j}, \quad z(0) = x(0) \tag{45}$$

where

$$\delta_i = \bar{u}_i(x) + \sum_{j=1}^{m} \sum_{k=1}^{m(m-1)/2} \left(\langle \nabla \beta_i^k, b_j \rangle \alpha_j^k - \langle \nabla \alpha_i^k, b_j \rangle \beta_j^k \right) sgn(j-i) \frac{1}{k}$$

$$Skew(\nu_{i,j}) = \sum_{k=1}^{m(m-1)/2} \left(\beta^k \wedge \alpha^k \right) \frac{1}{k}$$

and

$$\beta^k = (\beta_1^k \cdots \beta_m^k)^T, \quad \alpha_k = (\alpha_1^k, \cdots \alpha_m^k)^T.$$

Proof. Apply the control input (44) to system (29) we have

$$\dot{x} = \sum_{i=1}^{m} G_i(x) + \sqrt{\frac{2}{\epsilon}} \sum_{k=1}^{m(m-1)/2} R_k(x) \sin(\frac{kt}{\epsilon}) + Q_k(x) \cos(\frac{kt}{\epsilon})$$

where

$$\begin{aligned} G_i(x) &= b_i(x)\bar{u}_i(x) \\ R_k(x) &= \sum_{i=1}^{m} b_i(x)\alpha_i^k(x) \\ Q_k(x) &= \sum_{i=1}^{m} b_i(x)\beta_i^k(x) \end{aligned}$$

The desired result follows from the proof of Theorem 2.1 and the fact that

$$[b_1\alpha(x),\ b_2\beta(x)] = \alpha\beta[b_1, b_2] + b_2\langle\nabla\beta,\ b_1\rangle - b_1\langle\nabla\alpha, b_2\rangle.$$

□

3 Controllability and Holonomy

In this section we provide the motivation for using the control input in Algorithm 2.1.

Consider a system of the form

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= f_1(x_1, x_2)u_1 + f_2(x_1, x_2)u_2 \end{aligned}$$

where $(x_1, x_2, x_3)^T \in \Re^3$. Assume that f_1, f_2 are smooth functions and define by

$$g(x_1, x_2) = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}.$$

If there exists a ball $B_\epsilon \in \Re^2$ such that $g(x_1, x_2) \neq 0$ for all (x_1, x_2) inside the ball and zero outside the ball, then the system is controllable but not locally-locally controllable (see, e.g., [HH70]). To see this, let $x_0 = (x_1^0, x_2^0, x_3^0)$ and

$x_f = (x_1^f, x_2^f, x_3^f)$ be two given points and $(\hat{x}_1, \hat{x}_2) \in \Re^2$ a point such that $g(\hat{x}_1, \hat{x}_2) \neq 0$. Suppose that Ω is a closed ball in B_ϵ containing $(\hat{x}_1, \hat{x}_2)$ with boundary $\delta\Omega$, then by Green's Theorem we have that

$$\int_{\delta\Omega} f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2 = \iint_\Omega g(x_1, x_2)dx_1dx_2.$$

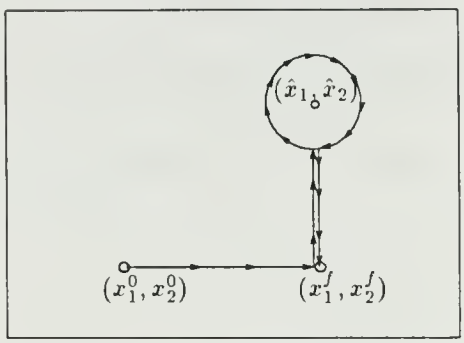

Figure 2: Holonomy of a control path in $\Re^2$.

A trajectory which links x_0 to x_f is shown in Figure 2, where control input $u(t) \in \Re^2, t \in [0, T_1]$, is chosen to steer the system from (x_1^0, x_2^0) to (x_1^f, x_2^f). Then, Ω is chosen so that

$$n = \left(x_3^f - x_3^0 - \int_0^{T_1} f_1 u_1 dt + f_2 u_2 dt \right) \left(\iint_\Omega g(x_1, x_2)dx_1dx_2 \right)^{-1}$$

is an integer (could be negative) and control input $u(t) \in \Re^2, t \in [T_1, T]$, is chosen to cause the system to encircle Ω n times, clockwise if n is negative. It is quite clear that the system is not locally-locally controllable for a point lying outside the B_ϵ ball. In order for it to be locally-locally controllable the set $\{(x_1, x_2) \in \Re^2, g(x_1, x_2) \neq 0\}$ has to be dense in $\Re^2$.

Generalizing from the above example, we have the following result.

Theorem 3.1 *Consider a nonlinear control system of the form*

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= f(x)u \end{aligned} \tag{46}$$

where $\{(x,y), x \in \Re^n, y \in \Re^m\}$ *is the state,* $u \in \Re^n$ *the control input, and* $f(x) = [f_1(x), \ldots, f_n(x)]$ *a collection of* x*-dependent* C^1 *vector fields. The system is controllable if and only if the span of*

$$\{g_{ij}(x), x \in \Re^n, 1 \leq i,j \leq n\}$$

is equal to $\Re^m$*, where*

$$g_{ij} = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}$$

In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.1 *(1) Let* $c : [0,1] \longrightarrow \Re^m, c(0) = 0$*, be a* C^1 *curve and associate with* c *the following grid*

$$G(c) = \{y \in \Re^m, y = \sum_{i=1}^{k} n_i c(t_i), n_i \in Z, t_i \in [0,1], i \in [1,k], 1 \leq k \leq \infty\}$$

we claim that $G(c)$ *is in fact a linear subspace of* $\Re^m$*.*

(2) Morcover,

$$G(c) = \left\{ \sum_{i=1}^{k} \pm n c(t_i), n \in N, t_i \in [0,1] \right\}$$

is a subspace of $\Re^m$ *of dimension at least* $k/2$*, where* Z *denotes the set of integers and* N *the set of nonnegative integers.*

Remark 3.1 The lemma does not hold for ∞-dimensional space. Consider, for example, the case $c(t) \in L_2[0,1]$ such that

$$c(t)(x) = \begin{cases} 1 - x/t, & 0 \leq x < t, \\ 0, & t \leq x \leq t, \end{cases}$$

for $t \neq 0$ and $c(0) = 0$. Then, $\tilde{c}(t) \triangleq c(t^2)$ is C^1 and $G(\tilde{c})$ is not a linear subspace, because for any $\mu \in G(\tilde{c})$ we have that

$$\mu(0) \in N.$$

□

Proof of Lemma 3.1. Since

$$c(t) = \int_0^t c'(\tau)d\tau$$

so we have that

$$G(c) \subseteq \text{span}\{c'(\tau), \tau \in [0,1]\} \triangleq L$$

We will show that

$$G(c) = L \tag{47}$$

Let l be the dimension of L and pick l points $\{\tau_i\}_{i=1}^l$ such that

$$\{c'(\tau_i), i \in [1,l]\}$$

forms a basis of L.

Define a map $C : \Re^l \longrightarrow L$ by

$$C(\delta_1, \ldots, \delta_l) = \sum_{i=1}^{l} c(\tau_i + \delta_i)$$

The directional derivative of C is

$$\frac{\partial C}{\partial \delta_i}|_{\delta_i=0} = c'(\tau_i)$$

and thus the Jacobian of C is

$$[c'(\tau_1), \ldots, c'(\tau_l)] \triangleq D$$

It is clear that D is nonsingular and by the rank Theorem, for sufficiently small δ, $C(B_\delta(0))$ has nonempty interior in L, where $B_\delta(0)$ is an open-ball of radius δ. Thus, by construction we have that

$$Im(C) \subset G(c)$$

On the other hand, $G(c)$ has the additional properties that

$$G(c) + G(c) \subset G(c)$$

and

$$kG(c) \subset G(c), \quad \forall k \in Z.$$

Using these facts we conclude that $G(c) = L$. □

Proof of Theorem 3.1. To show that the condition is necessary, suppose that

$$\text{Span}\{g_{ij}(x), x \in \Re^n, 1 \leq i, j \leq n\} \neq \Re^m$$

Then, there exists a vector $c \in \Re^m$ such that

$$\langle e, g_{ij} \rangle = 0, \ \forall 1 \leq i, j \leq n$$

Consider the differential equation for $\langle y, e \rangle$

$$\langle y, e \rangle' = \langle f(x)u, e \rangle = \sum_{i=1}^{n} \langle f_i(x), e \rangle \dot{x}_i = \sum_{i=1}^{n} \tilde{f}_i(x) \dot{x}_i,$$

where $\tilde{f}_i(x) = \sum_{j=1}^{m} f_i^j(x) e_j$.

Note that $\tilde{f}_i : \Re^n \longrightarrow \Re$ is a C^1 function. It is clear that

$$\frac{\partial \tilde{f}_i}{\partial x_j} - \frac{\partial \tilde{f}_j}{\partial x_i} = \langle e, g_{ij}(x) \rangle = 0, \ 1 \leq i, j \leq n$$

In other words, the integrability condition is satisfied and there exists a C^2 function g such that

$$\frac{\partial g}{\partial x_i} = \tilde{f}_i$$

Therefore, for any $x(0) = x(1)$ we have that

$$\langle y(0), e \rangle = \langle y(1), e \rangle$$

This contradicts the assumption that the system is controllable.

To show that the system is controllable at (x_0, y_0), choose m points $\{\bar{x}_k\}_{k=1}^{m}$ such that

$$\{g_{i_k j_k}(\bar{x}_k), 1 \leq k \leq m, 1 \leq i_k, j_k \leq n\} \triangleq F$$

forms a basis of $\Re^m$. We will construct a curve that takes the system from (x_0, y_0) to an arbitrary state (x, y) as follows:

First, denote by $S_k(\epsilon)$ a simple closed curve based at $\bar{x}_k$ and lying on the $x_{i_k} - x_{j_k}$ plane, e.g., circle or square,

$$\begin{aligned} S_k(\epsilon) &= \{(x^1, \ldots, x^n), x^i = \bar{x}_k^i, i \neq i_k, j_k, x^{i_k} \in [\bar{x}_k^{i_k}, \bar{x}_k^{i_k} + \epsilon], \\ &\quad x^{j_k} \in [\bar{x}_k^{j_k}, \bar{x}_k^{j_k} + \epsilon]\} \end{aligned}$$

Define a primitive curve $M(k, \epsilon, N)$ in the x or the control space to be a curve with the following three seqments:

1. The first segment links x_0 to $\bar{x}_k$.
2. The second segment traverses $S_k(\epsilon)$ N times, clockwise if $N > 0$ and counter clockwise if $N < 0$.
3. The last segment is the negative of segment 1, and thus links $\bar{x}_k$ back to x_0.

Let c_k be a curve of $\Re^m$ such that

$$c_k(\epsilon) = \iint_{S_k(\epsilon)} g_{i_k j_k}(x) dx_{i_k} dx_{j_k}, \quad c_k(0) = 0$$

Then, by Green's Theorem, the final destination of traversing the primitive curve $M(k, \epsilon, N)$ is

$$(x_0, y_0 + n c_k(\epsilon))$$

Therefore, if we use all possible motions of $M(k, \epsilon, N)$ we can achieve points of the form

$$\{(x_0, y) : y \in y_0 + \sum_{k=1}^{m} G(c_k)\}$$

By Lemma 3.1,

$$\sum_{k=1}^{m} G(c_k)$$

is a linear subspace of $\Re^m$, which contains the set

$$B \triangleq \{g_{i_k j_k}(\bar{x}_k)\epsilon^2 + o(\epsilon^2),\ 1 \leq k \leq m\}$$

But, by assumption B forms a basis of $\Re^m$ for sufficiently small ϵ. Therefore,

$$\sum_{k=1}^{m} G(c_k) = \Re^m$$

and

$$y_0 + \sum_{k=1}^{m} G(c_k) = \Re^m.$$

□

Using the constructive procedure in the proof of Theorem 3.1, we can obtain an upper bound for the optimal control input.

Proposition 3.1 *Consider the system*

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= f(x)u \end{aligned}$$

with boundary conditions $x(0) = x(1) = x_0$, $y(0) = y_0$ *and* $y(1) = y_f$. *Here* $(x, y) \in \Re^n \times \Re^m$. *Assume that there exist* m *points* $\{\bar{x}_k\}_{k=1}^m$ *such that*

$$\{g_{i_k j_k}(\bar{x}_k), 1 \leq k \leq m, 1 \leq i_k, j_k \leq n\}$$

forms a basis of $\Re^m$. *Then the optimal control input, which is piecewise smooth, is bounded above by*

$$\int_0^1 \|u(t)\|^2 dt \leq C_1 \|y_f - y_0\|^2 \left(C_2 \sum_{k=1}^{m} \|x_0 - \bar{x}_k\|^2 + C_3 \right) \tag{48}$$

where C_1, C_2 *and* C_2 *are constants.*

Proof. From the proof of Theorem 3.1, we know that there exists a ball of radius σ, $B_\sigma(0)$, which lies in the set

$$\left\{ \sum_{k=1}^{m} c_k(\epsilon_k^1) - c_k(\epsilon_k^2),\ 0 \leq \epsilon_k^i \leq \bar{\epsilon} \right\} \supset B_\sigma(0).$$

Therefore, we can express

$$y_f - y_0 = \eta \left(\sum_{k=1}^{m} c_k(\epsilon_k^1) - c_k(\epsilon_k^2) \right) \tag{49}$$

where

$$\eta = [\|y_f - y_0\|/\sigma] + 1, \quad \epsilon_k^i \in [0, \bar{\epsilon}].$$

and $[\|y_f - y_0\|/\sigma]$ denotes the integer part of $\|y_f - y_0\|/\sigma$.

First, construct control input $u(t), t \in [0,1]$, so that the resulting $x-$trajectory is the composition of the following primitive motions

$$M(1, \epsilon_1, 1), \cdots M(m, \epsilon_m, 1)$$

For example, the associated control input for $M(1, \epsilon_1, 1)$ has the form

$$u = \begin{cases} 3m(\bar{x}_1 - x_0), \ t \in [0, \frac{1}{3m}], \\ \tilde{u}((t - \frac{2}{3m})3m), \ t \in (\frac{1}{3m}, \frac{2}{3m}], \\ -3m(\bar{x}_1 - x_0), \ t \in (\frac{2}{3m}, \frac{1}{m}] \end{cases}$$

where

$$\tilde{u} = (0, \cdots u_{i_1}(t), 0 \cdots u_{j_1}(t), 0, \cdots 0)^T$$

and

$$u_{i_1}(t) = \begin{cases} \epsilon_1, \ t \in [0, \frac{1}{4}], \\ 0, \ t \in (\frac{1}{4}, \frac{1}{2}], \\ -\epsilon_1, \ t \in (\frac{1}{2}, \frac{3}{4}], \\ 0, \ t \in [\frac{3}{4}, 1] \end{cases} \qquad u_{j_1} = \begin{cases} 0, \ t \in [0, \frac{1}{4}], \\ \epsilon_1, \ t \in (\frac{1}{4}, \frac{1}{2}], \\ 0, \ t \in (\frac{1}{2}, \frac{3}{4}], \\ -\epsilon_1, \ t \in (\frac{3}{4}, 1] \end{cases}$$

The corresponding cost is computed to be

$$\int_0^{1/m} \langle u, u \rangle dt = \frac{2}{3m} \|x_0 - \bar{x}_1\|^2 + 3m\epsilon_1^2$$

and the total cost is

$$I_1 = \frac{2}{3m} \sum_{i=1}^{m} \|x_0 - \bar{x}_i\|^2 + 3m \sum_{i=1}^{m} \epsilon_i^2.$$

Then, using the above control input and time scaling to construct new control input so that the resulting x-trajectory is the composition of the primitive motions

$$M(1, \epsilon_1, \eta), \cdots M(m, \epsilon_m, \eta)$$

Obviously, the final input cost is given by

$$\eta^2 I_1$$

and from which we have

$$\eta^2 I_1 \leq \frac{\|y_f - y_0\|^2}{\sigma^2} \left(\frac{2}{3m} \sum_{i=1}^{m} \|x_0 - \bar{x}_i\|^2 + 3m \sum_{i=1}^{m} \epsilon_i^2 \right).$$

□

Proposition 3.2 *Under the same conditions as in Proposition 3.1, if we restrict that*

$$\|x(t)\| \leq \Omega$$

and assume that $\|y_f - y_0\|$ is sufficiently large, then the optimal control input

$$I_{op} = \int_0^1 \langle u,\ u \rangle dt$$

is bounded by the following inequalities

$$C_1 \|y_f - y_0\|^2 \leq I_{op} \leq C_2 \|y_f - y_0\|^2 \tag{50}$$

where C_1 and C_2 are some constants.

Proof. Because of Proposition 3.1 we only need to show the lower bound. The following inequalities follow from Schwartz inequality

$$\begin{aligned} \|y_f - y_0\| &= \| \int_0^1 f(x) u dt \| \\ &\leq \left(\int_0^1 \|f(x)\|^2 dt \right)^{1/2} \left(\int_0^1 \langle u,\ u \rangle \right)^{1/2} \\ &\leq \max_{\|x\| \leq \Omega} \|f(x)\| \cdot \left(\int_0^1 \langle u, u \rangle dt \right)^{1/2} \end{aligned}$$

Therefore, we have that

$$\int_0^1 \langle u, u \rangle dt \geq \frac{\|y_f - y_0\|^2}{\left(\max_{\|x\| \leq \Omega} \|f(x)\| \right)^2} \triangleq C_1 \|y_f - y_0\|^2.$$

□

From the proof of Proposition 3.1, we observe that a system of nonlinear equations, namely that defined by Eq. (49), needs to be solved if we use inputs of bounded quadratic cost. It is, however, possible to obtain asymptotic control input by solving only a system of linear equations if we allow the cost to be unbounded quadratic functions. We present such an algorithm below for system of the form

$$\begin{aligned} \dot{x} &= u, \ x \in \Re^n, \\ \dot{y} &= f(x)u, \ y \in \Re^m \end{aligned}$$

with boundary conditions

$$x(0) = x(1) = x_0, \ y(1) - y(0) = \delta.$$

Algorithm 3.1 **Step 1** Choose a small $\epsilon > 0$.

Step 2 Compute the vector $z = (z_1, \cdots z_m)^T$ by solving the following system of linear equations

$$\epsilon^2 Dz = \delta$$

where

$$D = \{g_{i_1,j_1}(\bar{x}_1), \cdots g_{i_m,j_m}(\bar{x}_m)\}$$

Step 3 Use the control input in the proof of Proposition 3.1 to cause the x-trajectory to follow the following primitive motions

$$M(1, \epsilon, [z_1]), M(2, \epsilon, [z_2]), \cdots M(m, \epsilon, [z_k])$$

where $[z_k]$ denotes the closest integer toz_k.

It is easy to see that the contribution of the above primitive motions in the y-space has the form

$$\tilde{\delta} = \sum_{k=1}^{m} [z_k] \left(g_{i_k,j_k} \epsilon^2 + O(\epsilon^2) \right)$$

where

$$g_{i_k,j_k}\epsilon^2 + O(\epsilon^2) = C_k(\epsilon) = \int\int_{S_k(\epsilon)} g_{i_k,j_k}(x)dx_{i_k}dx_{j_k}$$

The error term is bounded by

$$\|\tilde{\delta} - \delta\| \leq \|\tilde{\delta} - \bar{\delta}\| + \|\bar{\delta} - \delta\| \tag{51}$$

and

$$\bar{\delta} = \sum_{k=1}^{m} z_k \left(g_{i_k,j_k}\epsilon^2 + O(\epsilon^2)\right)$$

The first term in (51) is in the order of ϵ^2,

$$\|\tilde{\delta} - \bar{\delta}\| \sim \epsilon^2.$$

The second term is in the order of 1,

$$\begin{aligned} \bar{\delta} - \delta &= (\epsilon^2 D + O(\epsilon^2))\epsilon^{-2}D^{-1}\delta - \delta \\ &= ((D + O(1))D^{-1} - I)\delta \sim O(1). \end{aligned}$$

However, if $f(x)$ is twice continuously differentiable, then it is possible to show that $C_k(\epsilon) = g_{i_k,j_k}\epsilon^2 + O(\epsilon^3)$ and

$$\|\tilde{\delta} - \delta\| \sim \epsilon. \tag{52}$$

For example, if the primitive motions, $M(k, \epsilon, [z_k])$, are circles of radius ϵ with time interval $[0, 2\pi]$, then (52) holds. In this case, the control input has the form

$$\frac{1}{\epsilon}\sin(\frac{t}{\epsilon^2}), \frac{1}{\epsilon}\cos(\frac{t}{\epsilon^2}). \tag{53}$$

Remark 3.2 The results of the above algorithm provide the motivation for using highly oscillatory control inputs in Section 2. □

3.1 Controllability of Linearized Nonholonomic System

In this section, we discuss linearization of a nonholonomic system around a nominal trajectory and controllability properties of the linearized system.

Consider a nonholonomic system of the form

$$\dot{x} = A(x)u, \quad x \in \Re^n, \tag{54}$$

where $u = (u_1, \cdots u_m)^T$, and $A = (A_1, \cdots A_m)$.

Let $\bar{u}$ be a nominal control input to system (54) and δu a perturbation term. Apply the perturbed control input $u = \bar{u} + \delta u$ to (54) and linearize the system around the nominal trajectory $\bar{x}$, we obtain the following differential equation for the linearized system

$$\frac{d}{dt}(\delta x) = \left(\sum_{i=1}^{m} \frac{\partial A_i}{\partial x} u_i \right) \delta x + A(\bar{x}) \delta u. \tag{55}$$

where

$$x = \bar{x} + \delta x + o(\|\delta x\|)$$

Note that system (55) is linearized about a nominal trajectory, not about a fixed point as in usual practice.

We have the following results for controllability of the linearized system.

Theorem 3.2 *Assume that the columns of $A(x)$ are C^1 functions, and there exist constants C and B such that*

$$\|A(x)\| \leq C + B\|x\|$$

then system (54) is controllable if and only if for every x_0 there exists control input $u(\cdot)$ such that the linearized system (55) is controllable.

Remark 3.3 *The above theorem is used in [Cor91] to construct smooth, time-periodic feedbacks for stabilization of system (54).*

To illustrate Theorem 3.2, consider the the following simpler system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{y} &= f_1(x)u_1 + f_2(x)u_2,\ y \in \Re^m. \end{aligned} \tag{56}$$

Linearize the system about a nominal trajectory $(\bar{u}_1, \bar{u}_2) \in L_2[0,1]$, we have

$$\begin{aligned} \frac{d}{dt}(\delta x_1) &= \delta u_1 \\ \frac{d}{dt}(\delta x_2) &= \delta u_2 \\ \frac{d}{dt}(\delta y) &= \left(\sum_{i=1}^{2} \frac{\partial f_i}{\partial x_1}\bar{u}_i\right)\delta x_1 + \left(\sum_{i=1}^{2} \frac{\partial f_i}{\partial x_2}\bar{u}_i\right)\delta x_2 + \sum_{i=1}^{2} f_i(\bar{x})\delta u_i. \end{aligned} \tag{57}$$

It is easy to see that the linearized system is controllable if and only if it is possible to reach from the origin any state of the form $(0, 0, \delta y)$. Therefore, we can assume that

$$\int_0^1 \delta u_1 dt = \int_0^1 \delta u_2 dt = 0$$

Integrating the third equation of (57) by parts and using the following boundary conditions

$$\delta x_i(0) = \delta x_i(1) = 0, \quad i = 1, 2$$

yield

$$\begin{aligned} \delta y(1) &= \int_0^1 \left(\frac{\partial f_1}{\partial x_1}\bar{u}_1 + \frac{\partial f_2}{\partial x_1}\bar{u}_2\right)\delta x_1 dt + \int_0^1 \left(\frac{\partial f_1}{\partial x_2}\bar{u}_1 + \frac{\partial f_2}{\partial x_2}\bar{u}_2\right)\delta x_2 dt \\ &+ \int_0^1 (f_1(\bar{x})\delta u_1 + f_2(\bar{x})\delta u_2)\, dt \\ &= \int_0^1 \left(\frac{\partial f_1}{\partial x_1}\bar{u}_1 + \frac{\partial f_2}{\partial x_1}\bar{u}_2\right)\delta x_1 dt + \int_0^1 \left(\frac{\partial f_1}{\partial x_2}\bar{u}_1 + \frac{\partial f_2}{\partial x_2}\bar{u}_2\right)\delta x_2 dt \\ &+ \int_0^1 \left(f_1(\bar{x})\frac{d}{dt}(\delta x_1) + f_2(\bar{x})\frac{d}{dt}(\delta x_2)\right) dt \\ &= \int_0^1 \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)(\bar{u}_2\delta x_1 - \bar{u}_1\delta x_2) dt \end{aligned}$$

Assuming for simplicity that $(\bar{u}_1(t), \bar{u}_2(t))$, $t \in [0,1]$, are continuous functions, then we have the following criteria for controllability of the linearized system: *there exist* $t_1, \cdots t_m \in [0,1])$ *such that*

$$\begin{aligned} \bar{u}_1^2(t_i) + \bar{u}_2^2(t_i) &\neq 0 \\ \text{Span}\left(\left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right)(t_i),\ 1 \leq i \leq m\right) &= \Re^m. \end{aligned}$$

Note that we can use the above result to provide an alternative proof of Theorem 3.1.

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Appendix A: Comparision with Coron's Results

It is worthwhile to compare results of this paper with that of Coron's ([Cor91]). First, let's summarize several relevant results of this paper.

Consider a nonholonomic system of the form

$$\dot{x} = b_1(x)u_1 + \cdots b_m(x)u_m, \quad x \in \Re^n \tag{58}$$

and a holonomic system of the form

$$\dot{y} = g(y,t),\ y \in \Re^n,$$

Let l be the Lie-bracket order of (58) and fix a ball centered at x_0 and of radius δ, $B(x_0,\delta)$.

1. If $g(y,t) \in C^r$ for any t, then there exists feedback solution $u_\epsilon(x,t,\epsilon) \in C^{r-l}$ for any t and ϵ such that

$$\|x - y\|_{C[0,2\pi]} \to 0, \text{ as } \epsilon \to 0.$$

2. If $g(y,t)$ is 2π-periodic in t and $y(2\pi) \in B(x,\alpha\delta)$, $(0 < \alpha < 1)$, for any $y(0) \in B(x,\delta)$, then

$$\|x - y\|_{C[0,\infty)} \to 0, \text{ as } \epsilon \to 0.$$

3. If the holonomic system is exponentially stable, i.e.,

$$\|y(t) - x_0\| \le e^{-\alpha t}\|y(0) - x(0)\|$$

for any $y(0) \in B(x_0,\delta)$, then there exists an attractor A_ϵ such that

$$\text{Dist}(x, A_\epsilon) \le Ce^{-\alpha t}$$

and

$$\max_{z \in A_\epsilon} \|x_0 - z\| \to 0, \text{ as } \epsilon \to 0.$$

Recall that, for a first-order Lie bracket system where the nominal system is given by $\dot{y} = -y$, the control input has the form

$$u(x,t,\epsilon) = \alpha(x) + \sqrt{\frac{2}{\epsilon}} \sum_{i=1}^{m(m-1)/2} \beta_i(x)\sin(\frac{it}{\epsilon}) + \gamma_i(x)\cos(\frac{it}{\epsilon})$$

which is unbounded as $\epsilon \to 0$. To avoid using unbounded control we can rescale time with

$$v_\epsilon = \sqrt{\epsilon}u(x, t\sqrt{\epsilon}, \epsilon)$$

The trajectory of the system under the application of v_ϵ, denoted by $\tilde{x}(t)$, is related to $x(t)$ by

$$\tilde{x}(t) = x(t\sqrt{\epsilon})$$

and the rate of convergence to an attractor of radius $\sim \sqrt{\epsilon}$ is $e^{-t\sqrt{\epsilon}}$.

It is interesting to note that a similar rate of convergence is also observed in the solution of Riccati equation. Consider a discrete-time Reccati equation

$$P(n+1) = P(n)\left(1 - \frac{P(n)}{1+P(n)}\right) = \frac{P(n)}{1+P(n)},\ P(0) \geq 0$$

associated with solution of the following system

$$\begin{aligned} x(n+1) &= x(n) + u(n) \\ &\min \sum_{n=1}^{\infty} u(n)^2 \end{aligned}$$

It is clear that the solution $P(n) = \frac{P(0)}{1+nP(0)} \to 0$ as $n \to \infty$.

Consider the perturbed equation

$$P(n+1) = \epsilon + \frac{P(n)}{1+P(n)}$$

we have $P(n) \to \tilde{P} > 0$, where $\tilde{P} = \epsilon + \frac{\tilde{P}}{1+\tilde{P}}$. Thus, the equation for $\tilde{P}$ is

$$\tilde{P}^2 - \epsilon\tilde{P} - \epsilon = 0$$

which gives

$$\tilde{P} = \frac{\epsilon + \sqrt{\epsilon^2 + 4\epsilon}}{2} \sim \sqrt{\epsilon}.$$

Therefore,

$$P(n) - \tilde{P} \sim (\frac{1}{1+\tilde{P}})^{2n} \sim e^{-2\sqrt{\epsilon}n}.$$

We would like to ask the following question: *Can we have exponential stability of the goal point using the smooth time-periodic feedbacks given in our algorithm*? Note that in order for this to be true, the control input has to satisfy

$$u(x, 0, \epsilon) = 0$$

which implies that

$$\alpha(0) = 0, \gamma_i(0) = 0.$$

We will now show that the answer to the above question is negative[2]

Recall the Gershkovich inequalities ([VGS8]): If a system of the form $\dot{x} = B(x)u$, $x \in \Re^n$, $u \in \Re^m$, is regular in some neighborhood of x_0, $\eta_1(x_0) = m$, $\eta_i(x_0)$ is the dimension of the space V_i generated by Lie brackets of order less than i. Denote by $h(x_0) = \max_i\{\eta_i > \eta_{i-1}\} = \min\{i, \eta_i = n\}$ and define by $\phi(i) = j$ for $\eta_{j-1} < i \leq \eta_j$ ($\eta_0 = 0$), and some local coordinate system $\mathcal{X}_i : B(x_0, \sigma) \longrightarrow \Re$ parallelepiped

$$\Pi_{C,\epsilon}(x_0) = \{y \in B(x_0, \sigma) : |\mathcal{X}_i(y) - \mathcal{X}_i(x_0)| \leq C\epsilon^{\phi(i)}\}.$$

Also denote by $D_\epsilon(x_0)$ the set of all x_F such that

$$\dot{x} = B(x)u,$$

$x(0) = x_0$, $x(1) = x_F$ and $\int_0^1 \langle u, u\rangle^{\frac{1}{2}} dt \leq \epsilon$. Then there are local coordinate systems $\{\mathcal{X}_i\}$, positive C, c, ϵ_0, such that for $\epsilon < \epsilon_0$,

$$\Pi_{c,\epsilon}(x_0) \subset D_\epsilon(x_0) \subset \Pi_{C,\epsilon}(x_0).$$

[2]This result was first established in [FGL91b].

Notice that $\|x - x_0\|_* \triangleq \|x - x_0\|$ is equivalent to $\sum_{i=1}^{\eta} |\mathcal{X}_i(x) - \mathcal{X}_i(x_0)|$ for small enough $\|x - x_0\|$.

We need the following consequences of this fact: There exist $\alpha, \beta, \epsilon''$ such that

1. For any $y \in B(x_0, \epsilon'')$

$$\mathcal{J}(y) \leq \alpha \|y - x_0\|^{\frac{1}{\eta(x_0)}}.$$

2. There exists a sequence $y_k \longrightarrow x_0$, such that

$$\mathcal{J}(y_k) \geq \beta \|y_k - x_0\|^{\frac{1}{\eta(x_0)}}.$$

Here $\mathcal{J}(y)$ is the optimal cost for the problem $\dot{x} = B(x)u$, $x(0) = x_0$, $x(1) = y$, $\mathcal{J}(y) = \min_u \int_0^1 \langle u, u \rangle^{\frac{1}{2}} \mathrm{dt}$.

Definition 3.1 *The system $\dot{x} = B(x)u$ is $(\sigma, g(t))$ stabilizable at x_0 if and only if there exist positive k and Σ_0 such that for any $y \in B(x_0, \Sigma_0)$ there is a control $\tilde{u}(t), 0 \leq t < \infty$, satisfying the following condition*

$$\dot{x} = B(x)\tilde{u}(t), \quad x(0) = y,$$

$$\|\tilde{u}(t)\| \leq k \|x(t) - x_0\|^{\sigma}$$

and

$$\|x(t) - x_0\| \leq \|y - x_0\| g(t), \quad g(t) \geq 0.$$

Theorem 3.3 . *The system is not $(C, g(t))$ stabilizable at x_0 if $\int_0^{\infty} (g(t))^{\sigma} \mathrm{dt} < \infty$ and $\sigma > \frac{1}{\eta(x_0)}$.*

Proof: Let us pick a sequence $y_k \longrightarrow x_0$, $\mathcal{J}(y_k) \geq \frac{1}{c} \|y_k - x_0\|^{\frac{1}{\eta(x_0)}}$. The following "scaling symmetry" is well-known: If $\dot{x} = B(x)u$, $x(0) = x_0$, $x(T) = x_f$, $\mathcal{J}(x_0, x_f, T) = \min_u \int_0^T \langle u, u \rangle^{\frac{1}{2}} \mathrm{dt}$, then $\mathcal{J}(x_0, x_f, T) \equiv \mathcal{J}(x_0, x_f, 1)$ $(0 < T < \infty)$.

Let us construct on the time interval $[0, T + 1]$ the following control:

1. On $[0,T]$, "stabilizing" control $\tilde{u}(t)$, $\dot{x} = B(x)\tilde{u}$, $x(0) = y_k$.

2. On $[T,T+1]$, optimal control for the problem

$$\begin{aligned} \dot{y} &= B(y)u, \quad y(T) = x(T), \quad y(T+1) = x_0, \\ &\min \int_T^{T+1} \langle u,u\rangle^{\frac{1}{2}} \mathrm{dt} \end{aligned}$$

Then using the Gershkovich inequality and the property of optimality we have

$$\begin{aligned} \beta\|y_k - x_0\|^{\frac{1}{\eta(x_0)}} &\leq \mathcal{J}(y_k) \\ &\leq \int_0^T \|u(t)\| + \mathcal{J}(x(t)) \\ &\leq \beta \int_0^T \|y_k - x_0\|^{\sigma} g(t)^{\sigma} + \alpha g(T)^{\frac{1}{\eta(x_0)}} \|y_k - x_0\|^{\frac{1}{\eta(x_0)}}. \end{aligned}$$

Let $\sigma > \frac{1}{\eta(x_0)}$, $\int_0^{\infty} (g(t))^{\sigma}\ \mathrm{dt} = \mathcal{D} < \infty$. Then

$$\beta - \alpha g(T) \leq k\mathcal{D}\|y_k - x_0\|^{\sigma - \frac{1}{\eta(x_0)}},$$

for any k and T. Because for some sequence $g(T_m) \longrightarrow 0$, we obtain $\beta \leq 0$, a contradiction. □

Appendix B: Analytic Solutions of Generalized Brockett System

In this appendix, we study analytic solutions for the following generalized Brockett system

$$\begin{aligned} \dot{x} &= u, \\ \dot{y} &= \left(\sum_{i=1}^{n} A_i u_i\right) x \end{aligned} \tag{59}$$

where $x, u \in \Re^n$, $y \in \Re^m$, $A_i \in \Re^{m \times n}$ for $i = 1, \cdots n$, and boundary conditions are

$$x(2\pi) = x(0) = 0,\ y(2\pi) - y(0) = \delta = (\delta_1, \cdots \delta_m)^T.$$

If we use the following Fourier representation for the control input

$$u_i = \sum_{p=1}^{\infty} a_p^i \sin pt + \sum_{p=1}^{\infty} b_p^i \cos pt, \quad i = 1, \cdots n \tag{60}$$

then the cost to be minimized has the form

$$J = \int_0^{2\pi} \langle u, u \rangle dt = \sum_{p=1}^{\infty} \sum_{i=1}^{n} \left((a_p^i)^2 + (b_p^i)^2 \right)$$

Integrating the second equation of (59) by parts yields

$$\begin{aligned} \delta_i &= \int_0^{2\pi} \sum_{j=1}^{n} \sum_{k=1}^{n} (A_k(i,j) u_k)\, x_j dt \\ &= \int_0^{2\pi} \sum_{j=1}^{n} \sum_{k=1}^{n} A_k(i,j) \dot{x}_k x_j dt \\ &= \sum_{k<j} (A_k(i,j) - A_j(i,k)) \int_o^{2\pi} \dot{x}_k x_j dt \\ &= \pi \sum_{k<j} (A_k(i,j) - A_j(i,k)) \sum_{p=1}^{\infty} \frac{1}{p} (a_p^k b_p^j - a_p^j b_p^k) \end{aligned}$$

Denote by

$$X_p^1 = (a_p^1, \cdots a_p^n)^T, \quad X_p^2 = (b_p^1, \cdots b_p^n)^T$$

and define a linear operator

$$D(i,(k,j)) = A_k(i,j) - A_j(i,k),\ k < j, \tag{61}$$

from the space of $n \times n$ skew-symmetric matrices, denoted by $so(n)$, to $\Re^m$. Then, the expression for δ can be written more compactly as

$$\delta = \pi D(\sum_{p=1}^{\infty} \frac{1}{p} X_p^1 \wedge X_p^2). \tag{62}$$

It is not difficult to see that the necessary and sufficient condition for controllability of system (59) is that the linear operator D has rank m.

In the case that $m = n(n-1)/2$ we can identify $\Re^m$ with $so(n)$. If D is the identity operator, then system (59) reduces to the Brockett's canonical system

$$\begin{aligned} \dot{x} &= u, \\ \dot{Z} &= x \wedge u,\ Z \in so(n). \end{aligned}$$

Without loss of generality, we will therefore consider optimal solutions for systems of the form

$$\begin{aligned} \dot{x} &= u,\quad x(0) = x(2\pi) \in \Re^n \\ \dot{y} &= D(x \wedge u),\ y(2\pi) - y(0) = \delta \in \Re^m \end{aligned} \tag{63}$$

where the control input has the representation given by (60) and the cost to be minimized is

$$\int_0^{2\pi} \langle u,\ u \rangle dt = \sum_{p=1}^{\infty} \|X_p^1\|^2 + \|X_p^2\|^2$$

In order to describe the set of optimal solutions for the problem, we introduce the following geometric construction. First, denote by

$$J_n = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

the canonical symplectic matrix of $\Re^{2n}$, and by $Sp(n)$ the set of representations of the form

$$J_n = \sum_{i=1}^{n} x_i^1 \wedge x_i^2$$

where $\{x_i^l, 1 \leq i \leq n, 1 \leq l \leq 2\}$ is an orthonormal basis of $\Re^{2n}$. Then, $Sp(n)$ has the following decomposition

$$Sp(n) = S_{2n} \times S_{2(n-1)} \times \cdots \times S_4 \times S_2 \tag{64}$$

where S_k is the unit sphere of $\Re^k$. Consider, for example, the case $n = 2$. Let $\{x_i^1, x_i^2\}_{i=1}^2$ be an orthonormal basis of $\Re^4$. Then,

$$J_2 = x_i^1 \wedge x_1^2 + x_2^1 \wedge x_2^2$$

and the linear subspace L_i defined by

$$L_i = \text{Span}\{x_i^1, x_i^2\}_{i=1}^2$$

is invariant under J_2, i.e., $J_2(L_i) \subset L_i$ because

$$J_2(x_i^1) = -x_i^2$$

and

$$J_2(x_i^2) = x_i^1.$$

Using these properties we can construct a decomposition of $Sp(2)$ as follows: First, pick an arbitrary $x_1^2 \in S_4$ and compute

$$x_1^1 = J_2 x_1^2$$

Then, compute an orthogonal complement of Span $\{x_1^1, x_1^2\}$

$$L_2 = R^4 \ominus \text{Span}\{x_1^1, x_1^2\} \sim R^2.$$

Finally, pick an arbitrary $x_2^2 \in S_2$ and compute

$$x_2^1 = J_2 x_2^2$$

As a result we have that $Sp(2) = S_4 \times S_2$.

Theorem 3.4 *Consider system (63). Suppose that there exists a unique* $S \in so(n)$ *such that* $\delta = \pi D(S)$[3]

$$S = \alpha_1 J_{n_1} \oplus \alpha_2 J_{n_2} \cdots \oplus \alpha_k J_{n_k} \oplus 0,\ \alpha_1 \geq \alpha_2 \cdots \geq \alpha_k \geq 0$$

and

$$R^n = Z_{n_1} \oplus Z_{n_2} \cdots \oplus Z_{n_k} \oplus Ker(S)$$

with $Dim(Z_{n_i}) = 2n_i$ *and* J_{n_i} *an operator in* Z_{n_i}, *then the set of optimal control to system (63) has the form*

$$\begin{aligned} u &= \sum_{p=1}^{rank(S)/2} X_p^1 \sin pt + X_p^2 \cos pt \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{i,j}^1 \sin(n_1 + \cdots + n_{i-1} + j) + X_{i,j}^2 \cos(n_1 + \cdots n_{i-1} + j) \end{aligned}$$

where

$$X_{i,j}^l = \sqrt{\alpha_i (n_1 + \cdots + n_{i-1} + j)} Y_{i,j}^l \in Z_{n_i},\ l = 1, 2$$

and

$$\{Y_{i,j}^1, Y_{i,j}^2, 1 \leq j \leq n_i\} \in Sp(n_i)$$

and the optimal cost is given by

$$I_{opt} = 2 \sum_{i=1}^{k} \alpha_i (n_1 + \cdots + n_{i-1} + \frac{1 + n_i}{2}) n_i$$

The above theorem is a special case of the following result.

Proposition 3.3 *Let* $S = \alpha_1 J_{n_1} \oplus \cdots \oplus \alpha_k J_{n_k} \oplus 0$, $\alpha_1 \geq \cdots \alpha_k \geq 0$, *and consider the following optimization problem: Find* $\{x_i^1, x_i^2\}$ *such that the constraint*

$$S = \sum_{i=1}^{\infty} x_i^1 \wedge x_i^2$$

[3]This is equivalent to $\delta \in Im(D)$ and $Ker(D) = 0$.

is satisfied and the cost function

$$I(x) = \sum_{i=1}^{\infty} C_i(\|x_i^1\|^2 + \|x_i^2\|^2),\ 0 < C_i < C_{i+1} \to \infty.$$

is minimized.

For any solution of the problem we have that

$$x_n^l = Y_{i,j}^l \sqrt{\alpha_i}$$

where $\{Y_{i,j}^l, 1 \le l \le 2, 1 \le j \le n_i\} \subset Sp(n_i)$,

$$n_1 + \cdots + n_{i-1} < n < n_1 + \cdots + n_i$$

and the optimal cost is

$$2\sum_{i=1}^{k} \alpha_i \sum_{j=n_1+\cdots+n_{i-1}+1}^{n_1+\cdots+n_i} C_j.$$

To prove Proposition 3.3, we need the following lemma.

Lemma 3.2 *Let* $\{x_1^1, x_1^2, x_2^1, x_2^2\}$ *be a solution to the following optimization problem:*

$$S = x_1^1 \wedge x_1^2 + x_2^1 \wedge x_2^2$$

and cost function

$$C_1\left(\|x_1\|^2 + \|x_1^2\|^2\right) + C_2\left(\|x_2^1 + \|x_2^2\|^2\right),\ 0 < C_1 < C_2$$

Then,

$$\|x_i^1\| = \|x_i^2\|, x_i^k \perp x_j^l \text{ for } (i,k) \neq (j,l)$$

Proof of Lemma 3.2. First, let us show that $x_i^1 \perp x_i^2$. Suppose that

$$x_i^2 = \mu x_i^1 + \gamma$$

Then, $\langle \gamma, x_i^1 \rangle = 0$, $\|\gamma\| < \|x_i^2\|$, $x_i^1 \wedge x_i^2 = x_i^1 \wedge \gamma$, and

$$C_1\left(\|x_i^1\|^2 + \|\gamma\|^2\right) < C_1\left(\|x_i^1\|^2 + \|x_i^2\|^2\right)$$

resulting in a contradiction.

Suppose that $\alpha = \|x_i^1\| \neq \|x_i^2\| = \beta$. Then, we can decrease $\|x_i^1\|^2 + \|x_i^2\|^2$ without changing $(x_i^1 \wedge x_i^2)$ by redefining

$$\tilde{x}_i^1 = \sqrt{\alpha\beta} x_i^1, \tilde{x}_i^2 = \sqrt{\alpha\beta} x_i^2$$

It remains to show that $x_1^k \perp x_2^l, k, l \in \{1, 2\}$. Since

$$\text{Rank}(S) \leq \text{Rank}(x_1^1 \wedge x_1^2) + \text{Rank}(x_2^1 \wedge x_2^2) = 4$$

we can restrict the problem to $\Re^4$. In this case,

$$S = \alpha_1 J_2 + \alpha_2 J_2, \ \alpha_1 \geq \alpha_2 \geq 0.$$

There are two possibilities: (i) $\alpha_2 = 0$, then $S = x \wedge y$ and $\|x\|^2 = \|y\|^2 = \alpha_1$. Let us show that $x_1^1 = x, x_1^2 = y$ and $x_2^1 = x_2^2 = 0$. If not, then we have that

$$C_1 \|x_1^1\|^2 + C_2 \|x_2^1\|^2 > C_1 \left(\|x_1^1\|^2 + \|x_2^1\|^2 \right) \geq \alpha_1 C_1$$

But, if $(x_1^1 = x, x_1^2 = y, x_2^1 = x_2^2 = 0)$, then $S = x_1^1 \wedge x_1^2$ and

$$C_1 \|x_1^1\|^2 + C_2 \|x_2^1\|^2 = C_1 \alpha_1$$

(ii) $\alpha_2 \neq 0$. Let us show that

$$c = \min(\|x_1^1\|^2, \|x_2^1\|^2) \geq \alpha_2.$$

If not, then

$$\| \left(S - x_1^1 \wedge x_1^2 \right) x \| \geq (\alpha_2 - c) \|x\|$$

and $\text{Rank}(x_2^1 \wedge x_2^2) = 4$, results in a contradiction.

Suppose that

$$L_2^{\perp} \triangleq (\text{Span}(x_2^1, x_2^2))^{\perp} \neq \text{Span}(x_1^1, x_1^2) \triangleq L_1$$

we can show that in this case $c > \alpha_2$. If not, for instance $\|x_1^1\| = \alpha_2$, pick $y \in (\text{Span}(x_2^1, x_2^2))^{\perp}$ and $y \notin \text{Span}(x_1^1, x_1^2)$. Then,

$$S(y) = (x_1^1 \wedge x_2^1)(z), \ \|z\| \leq \|y\|$$

where z is the projection of y onto L_1, and $\|S(y)\| < \alpha_2\|y\|$. But, for any vector $z \in \Re^1$ we have $\|S(z)\| \geq \alpha_2\|z\|$, a contradiction.

Thus, we have the following system of linear inequalities for $A_1 \triangleq \|x_1^1\|^2$ and $A_2 \triangleq \|x_2^1\|^2$

$$A_1 + A_2 \geq \alpha_1 + \alpha_2, (\alpha_2 \leq \alpha_1) \text{ (because of Claim 2)}$$

and

$$A_1 > \alpha_2, A_2 > \alpha_1, A_1 \geq A_2$$

Consequently, we obtain

$$C_1 A_2 + C_2 A_1 \geq C_1\alpha_1 + C_2\alpha_2$$

which contradicts the optimality condition. □

Claim 1: Let $A_0 = A_1 + A_2$, where A_i's are $n \times n$ square matrices with singular values $\{\lambda_i^l, 1 \leq i \leq n, 1 \leq l \leq 2\}$. Then,

$$\left(\sum_{i=1}^{n}(\lambda_i^l)^p\right)^{1/p} \leq \sum_{i=1}^{2}\left(\sum_{j=1}^{n}(\lambda_j^l)^p\right)^{1/p}, \; p \geq 1.$$

Claim 2: Let $S = \alpha_i J_{n_1} \oplus \cdots \oplus \alpha_k J_{n_k} \oplus 0$, $S = \sum_n x_n^1 \wedge x_n^2$, $\|x_n^1\| = \|x_n^2\|, \langle x_n^1, x_n^2\rangle = 0$, then

$$\sum_{i=1}^{k}\alpha_i n_i \leq \sum_n \|x_n^1\|^2.$$

Proof of Proposition 3.3. First, let us prove the existence of optimal solutions. For this consider a sequence of near-optimal solutions $\{x_m\} \triangleq \{x_{m,n}^l, 1 \leq l \leq 2\}, 1 \leq m \leq \infty$ such that

$$S = \sum_{n=1}^{\infty} x_{m,n}^1 \wedge x_{m,n}^2, \; \min I(x_n) = \inf_x I(x).$$

Because $C_i \to \infty$ the set $\{x_m\}$ is compact in the strong topology of l_2. Thus, there exist a subsequence

$$x_{m_k} \to \bar{x} = \{x_n^l, 1 \leq n < \infty, 1 \leq l \leq 2\}$$

such that $S = \sum_{n=1} x_n^1 \wedge x_n^2$ and

$$\sum_{n=1}^{\infty} \|x_n^1\|^2 + \|x_n^2\|^2 = \inf_x I(x) = \min_x I(x)$$

Therefore, let $\bar{x} = \{x_n^l, 1 \leq n \leq \infty, 1 \leq l \leq 2\}$ be an optimal solution. From the Lemma we have that

$$x_n^1 = x_n^2, x_n^k \perp x_n^m, \text{ for } (n,l) \neq (k,m),$$

and there exists a map

$$\pi : [1, \sum_{i=1}^{k} n_i] \to N$$

such that

$$\|x_n^1\|^2 = \begin{cases} 0, n \notin Im(D) \\ \alpha_k, \ n = \pi(i), \ n_1 + \cdots + n_{k-1} < i \leq n_1 + \cdots + n_k \end{cases}$$

To complete the proof, recall the following inequality: If $\alpha_1 \leq \alpha_2 \cdots \leq \alpha_n$, and $b_1 \geq b_2 \cdots \geq b_n$, then for any permutation π

$$\sum_{i=1}^{n} \alpha_i b_{\pi(i)} \geq \sum_{i=1}^{n} \alpha_i b_i.$$

□

Remark 3.4 Let $S = \alpha_i J_{n_1} \oplus \cdots \oplus \alpha_k J_{n_k} \oplus 0$, and $\alpha_1 \geq \alpha_2 \cdots \geq \alpha_k > 0$. Define the following functional on $so(n)$

$$f(S) = \sum_i \alpha_i i$$

If $\text{Ker}(D) \neq 0$, then the optimal control problem for system (63) is equivalent to

$$\min_S \{f(S) | D(S) = y(2\pi) - y(0)\}$$

We will give an analytic solution to this problem for $y \in \Re$:

$$S = \sum_{i\geq 1}^{k} \alpha_i (x_i^1 \wedge x_i^2),\ \alpha_1 \geq \alpha_2 \cdots \geq \alpha_k \geq 0 \tag{65}$$

and $x_i^k \perp x_j^l$ if $(i,k) \neq (j,l), \|x_i^j\| = 1$,

$$\min\{f(S) = \sum_{i=0}^{k} \alpha_i C_i\},\ 0 < C_1 < \cdots C_k, D(S) = \delta \in \Re.$$

Note that if $D : so(n) \to \Re$ is a linear functional then there exists unique $\tilde{S} \in SO(n)$ such that

$$D(S) = \mathrm{tr}(S\tilde{S})$$

where $tr(S)$ stands for the trace of S.

Claim 3: (1) Suppose that

$$S = \sum_{i\geq 1}^{k} \alpha_i (x_i^1 \wedge x_i^2),\ \ \alpha_1 \geq \alpha_2 \cdots \geq \alpha_k \geq 0$$

where $\|x_i^k\| = 1$ and $x_i^k \perp x_j^l$ for $(i,k) \neq (j,l)$, and

$$\tilde{S} = \sum_{i\geq 1}^{k} \beta_i (z_i^1 \wedge z_i^2),\ \ \beta_1 \geq \beta_2 \cdots \geq \beta_k \geq 0$$

where $\|z_i^k\| = 1$ and $z_i^k \perp z_j^l$ for $(i,k) \neq (j,l)$. Then,

$$|\delta| = |\mathrm{tr}(S\tilde{S})| \leq \sum_{i=1}^{k} \alpha_i \beta_i$$

(2)

$$\sum_{i=1}^{k} \alpha_i C_i \geq \frac{|\delta|}{\beta_1} C_1.$$

Proof of Claim 3. First, note that

$$|tr(S\tilde{S})| = |\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i \beta_j tr((x_i^1 \wedge x_i^2)(z_j^1 \wedge z_j^2))| \leq \sum_{i=1}^{k} \alpha_i \beta_j \Lambda ij$$

where

$$\begin{aligned}\Lambda ij &= |tr((x_i^1 \wedge x_i^2)(z_j^1 \wedge z_j^2))| \\ &= |\langle x_i^1, z_j^2\rangle\langle x_i^2, z_j^1\rangle - \langle x_i^1, z_j^1\rangle\langle x_i^2, z_j^2\rangle)| \\ &\leq \left(\langle x_i^1, z_j^1\rangle^2 + \langle x_i^1, z_j^2\rangle^2 + \langle x_i^2, z_j^1\rangle^2 + \langle x_i^2, z_j^2\rangle^2\right)/2\end{aligned}$$

Using orthonormality condition, we have

$$\sum_{j=1}^{k} \Lambda ij \leq \frac{\|x_i^1\|^2 + \|x_i^2\|^2}{2} = 1.$$

$$\sum_{i=1}^{k} \Lambda ij \leq \frac{\|z_i^1\|^2 + \|z_i^2\|^2}{2} = 1.$$

Therefore, $\{\Lambda_{ij}\}$ is a semibystochastic matrix and

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\beta_j\Lambda_{ij} = \sum_{\pi} \gamma_\pi \left(\sum_{i=1}^{k} \alpha_i\beta_{\pi(i)}\right)$$

where π is a permutation, $\gamma_\pi \geq 0, \sum_\pi \gamma(\pi) \leq 1$. But, for any permutation π we have that

$$\sum_{i=1}^{k} \alpha_i\beta_{\pi(i)} \leq \sum_{i=1}^{k} \alpha_i\beta_i.$$

To show the second part, observe that

$$\begin{aligned}\sum_{i=1}^{k} \alpha_i C_i &= \sum_{i=1}^{k} \alpha_i\beta_i\frac{C_i}{\beta_i} \geq \min_i(\frac{C_i}{\beta_i})\sum_{i=1}^{k} \alpha_i\beta_i \\ &= \frac{C_1}{\beta_1}\sum_{i=1}^{k} \alpha_i\beta_i \geq \frac{C_1}{\beta_1}|\delta|\end{aligned}$$

□

Proposition 3.4 *Suppose that*

$$\tilde{S} = \beta_1 J_{n_1} \oplus \cdots \oplus \beta_k J_{n_k} \oplus 0, \;\; \beta_1 \geq \beta_2 \geq \cdots \geq \beta_k \geq 0 \in so(n)$$

and

$$\Re^n = Z_{n_1} \oplus \cdots \oplus Z_{n_k} \oplus Ker(\tilde{S}).$$

Then, the set of optimal solutions of problem (65) has the form

$$\left\{ \frac{\delta}{\beta_1} (Z \wedge -S(Z)), \|Z\| = 1, Z \in Z_{n_1} \right\}$$

and

$$\min f(S) = \frac{|\delta|}{\beta_1} C_1$$

The proof is a direct application of Claim 3.

Remark 3.5 Proposition 3.3 is correct even if $C_i \not\to \infty$. Necessary and sufficient condition for the existence of solutions is: There exist $C_{\pi(1)}, \cdots C_{\pi(k)}$ such that

$$0 < C_{\pi(1)} \leq \cdots \leq C_{\pi(k)} \leq C_n, n \geq 1$$

where $k = Rank(S)/2$.

Claim 4: Consider optimal control of the following system

$$\dot{x} = B(x)u, \; x \in \Re^n, u \in \Re^m$$

with initial condition $x(0) = x_0, x(t) = x_f$ and cost

$$\inf \int_0^t \langle u, u \rangle dt = I_t(x_0, x_f)$$

Then, I_t satisfies the following triangle inequality

$$\sqrt{I_t(x_0, x_f)} \leq \sqrt{I_t(x_0, z)} + \sqrt{I_t(z, x_f)}$$

The proof follows from the time symmetry of the cost functional, i.e., $I_t(x_0, x_f) = \frac{1}{t} I_1(x_0, x_f)$.

Consider now Brockett's system

$$\begin{aligned} \dot{x} &= u \\ \dot{Y} &= \frac{1}{\pi}(x \wedge u), Y \in so(n) \end{aligned}$$

with boundary conditions $x(0) = x(2\pi) = 0$, and $Y(0) = 0, Y(2\pi) = S$, and cost $I = \int_0^2 \langle u, u \rangle dt$. Then, we have that $f(S) = I_{2\pi}((0,0),(0,S))$, and the following inequalities follow from the claim.

$$\sqrt{f(S_1 + S_2)} \leq \sqrt{f(S_1)} + \sqrt{f(S_2)}, \quad S \in so(n). \tag{66}$$

For a symmetric matrix $A = A^*$, we can also define the analog of the functional f: Let $\sigma(A) = (a_1, a_2, \cdots a_n)$, $|a_1| \geq |a_2| \geq \cdots \geq |a_n|$, be the spectrum values of A and define

$$\tilde{f}(A) = \sum |a_i| c_i$$

According to [Fri81], this functional is concave on the convex set of nonnegative matrices.

Corollary 3.1 *If $c_i = i$, then the following inequality holds*

$$\sqrt{\tilde{f}(A_1 + A_2)} \leq \sqrt{\tilde{f}(A_1)} + \sqrt{\tilde{f}(A_2)}$$

Proof: Associate with A_i the following skew-symmetric matrix

$$S_i = \begin{bmatrix} 0 & A_i \\ -A_i & 0 \end{bmatrix}$$

and apply inequality (66). □

In what follows we will give a *control system model* of Proposition 3.3. Let

$$K : L_2[0,1] \to L_2[0,1]$$

be a bounded linear operator, and suppose that

$$K - K^* = \sum_{i=1}^{\infty} \frac{1}{c_i} x_i^1 \wedge x_i^2, \; x_i^k \in L_2[0,1]$$

where $\|x_i^k\| = 1, x_i^k \perp x_j^l$ if $(i,k) \neq (j,l)$ and $c_i \geq \epsilon > 0$. Therefore,

$$(K - K^*)(v) = \sum \frac{1}{c_i} x_i^1 \langle v, x_i^2 \rangle - x_1^2 \langle v, x_i^1 \rangle, v \in L_2[0,1]$$

If $u = (u_1 . \cdots u_m)^T$ is a vector-valued function, we then define

$$K(u) = (K(u_1), \cdots K(u_m))^T$$

Consider nonlinear control problem for the following system

$$\dot{y} = \left(\sum_{i=1}^{n} A_i u_i \right) Ku, \; y \in \Re^m \tag{67}$$

with boundary condition $y(1) - y(0) = \delta$ and cost to be minimized

$$\min \int_0^1 \langle u, \; u \rangle dt.$$

It is possible to show that an optimal solution of the above system has to satisfy

$$u(t) = \sum_{n=1}^{\infty} X_n^1 x_n^1(t) + X_n^2 x_n^2(t)$$

$$y(1) - y(0) = D \sum_{n=1}^{\infty} \frac{1}{c_n} (X_n^1 \wedge X_n^2),$$

where $D : so(n) \rightarrow \Re^m$ is the same operator as defined in (61). We will call systems of this kind *integral nonholonomic systems*, in contrast to the usual differential nonholonomic systems.

This problem can be solved using previous results, and also note that system (67) is controllable if and only if

$$\max_{y \in \Re^m} \min_{D(S)=y} \text{Rank}(S) \leq \text{Rank}(K - K^*).$$

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